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## Abstract

## Full Text

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# INTEGRAL REPRESENTATIONS

*(Presented by Academician M. A. Lavrent' ev on 21 XI 1959)*

In preceding papers ([1−3]) I constructed the theory of integral representations for functions analytic in such bicircular domains $D$ at every boundary point of which the Levi determinant $L(\Phi) > 0$, where the boundary is given by

$$|z| - \varphi(|w|) \equiv \Phi(w, \overline{w}, z, \overline{z}) = 0,$$

i.e., for functions regular in bicircular domains whose boundaries are nowhere an analytic hypersurface. In the general case, as is known, $\varphi(r_1)$ may be constant on an interval beginning with the value $r_1 = 0$ ($r_1$ and $r_2 = \varphi(r_1)$ are conjugate radii of convergence) ([4]), and therefore this portion of the boundary of the domain $D$ may be an analytic hypersurface. Considering the boundary of the domain $D$ as given in the form $|w| = \psi(|z|)$, we conclude that a portion of the boundary of the domain $D$ beginning with the value $|z| = r_2 = 0$ may also be an analytic hypersurface. Thus, in the more general case, on the portions $0 \le |w| \le r_1$, $|z| = R_2$, and $|w| = R_1$, $0 \le |z| \le r_2$, one may have $L(\Phi) = 0$, and only outside them $L(\Phi) > 0$ (we leave aside the case in which $L(\Phi)$ vanishes at isolated points of the boundary of the domain $D$).

However, the integral representations of both kinds ([3]) remain valid for this general case if $r_1(\tau)$ and $r_2(\tau)$ are defined in the following way. The function $r_1(\tau)$, positive and continuously differentiable on the segment $0 \le \tau \le 1$, satisfies the conditions: $r_1(0) > 0$; in the interval $0 < \tau < 1$, $r_1'(\tau) > 0$, and $r_1'(1) = 0$. The function $r_2(\tau)$ is defined in terms of $r_1(\tau)$ in the same way as before:

$$r_2(\tau) = R_2 \exp\left[-\int_0^\tau \frac{\tau}{1-\tau}\, d\ln r_1'(\tau)\right], \tag{1}$$

where $R_2$ is a positive constant.

Indeed, since $r_1'(1) = 0$, it may happen that $r_2(1) > 0$ ([1]). For example,

$$r_1(\tau) = 2 - (1-\tau)^2.$$

Then

$$r_2(\tau) = [2 - (1-\tau)^2]\left(\frac{\sqrt{2} + (1-\tau)}{\sqrt{2} - (1-\tau)}\right)^{1/\sqrt{2}} (\sqrt{2}-1)^{\sqrt{2}},$$

and we have $r_1(0) = 1,\ r_1'(1) = 0,\ r_2(1) = 2(\sqrt{2}-1)^{\sqrt{2}}$. Taking into account that the monomials $w^m z^n$, $m \geq 0$, $n \geq 0$, are invariants with respect to the transformation

$$\frac{1}{2\pi}\int_0^{2\pi} dt \int_0^1 \frac{d}{du}\left[(u(r_1)u)^m (r_2(\tau)v)^n\right] d\tau = w^m z^n$$

for arbitrary continuous functions $r_1(\tau), r_2(\tau)$ ([5]), and $r_2(\tau)$ is defined in the same way as before, we arrive at the conclusion that in this general case as well, taking into account $r_1(0) > 0,\ r_2(1) > 0$, the entire preceding theory of integral representations is preserved.

Thus, the integral representations

$$F(w,z) = \frac{1}{4\pi^2 i}\int_0^{2\pi} dt \int_0^1 d\tau \int_{|\zeta|=1} \frac{\Phi\left[r_1(\tau)\zeta^n,\ r_2(\tau)\eta^n\right]}{\zeta - u}\, d\zeta, \qquad (2)$$

$$F(w,z) = \frac{1}{4\pi^2 i}\int_0^{2\pi} dt \int_0^1 d\tau \int_{|\zeta|=1} \frac{\zeta F\left[r_1(\tau)\zeta^n,\ r_2(\tau)\eta^n\right]}{(\zeta - u)^2}\, d\zeta, \qquad (3)$$

where

$$u = \tau\left(\frac{w}{r_1(\tau)}\right)^{1/n} + (1-\tau)\left(\frac{z}{r_2(\tau)}\right)^{1/n} e^{it},$$

$$\Phi(w,z) = F(w,z) + nwF_w'(w,z) + nzF_z'(w,z),$$

$n$ is the least integer not smaller than

$$\sup_{0<\tau<1} \frac{d\ln r_1(\tau)}{d\ln\tau},$$

hold in the domain

$$D:\ |w| < r_1(\tau),\ |z| < r_2(\tau),\ 0 \leq \tau \leq 1,$$

i.e., in the domain bounded by three hypersurfaces:

$$0 \leq |w| \leq r_1(0),\ |z| = r_2(0); \quad |w| = r_1(\tau),\ |z| = r_2(\tau),\ 0 < \tau < 1; \quad |w| = r_1(1),\ 0 \leq |z| \leq r_2(1).$$

Here, as we see, there appears a characteristic feature of analytic functions of two complex variables: the values of the function $F(w,z)$ in the domain $D$ are determined by its behavior, or by $\Phi(w,z)$, only on that part of the boundary of the domain $\overline{D}$ which is a nonanalytic hypersurface.

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## CITED LITERATURE

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*Note: Figure translations are in progress. See original paper for figures.*