Mathematics

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1960

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Abstract

Full Text

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ON A SPECIAL INTEGRAL EQUATION WITH A SHIFT

(Presented by Academician P. Ya. Kochina, 1 VI 1960)

Consider the integral equation

$$a(t)f(t) + \frac{b(t)}{\pi i}\int_L \frac{f(\tau)\,d\tau}{\tau - \alpha(t)} = h(t), \tag{1}$$

in which the functions $a(t)$, $b(t)$, $h(t)$ satisfy a Hölder condition on the closed Lyapunov contour L, with $a(t) \neq 0$ and $b(t) \neq 0$ at the points of L; the function $\alpha(t)$ maps the contour L one-to-one onto itself and has a derivative $\alpha'(t)$ distinct from zero and satisfying a Hölder condition on L; the solution $f(t)$ is sought in the class of functions satisfying a Hölder condition.

Special integral equations with kernels containing shifts, as is known, arise in the solution of boundary-value problems for differential equations of elliptic-hyperbolic (mixed) type. This above all explains the considerable attention that has been paid in recent years to the study of such equations ([1−6]).

The types of special equations considered in works ([1−6]) are essentially reduced to the Riemann boundary-value problem, and consequently solutions in closed form have been found for them. The special integral equation with shift in the form (1) is reduced to a boundary-value problem for analytic functions that is more general than the Riemann problem.

In general, the solution of equations of the form (1) will not have a closed form.

In the present work, the integral equation (1) is investigated under the following assumptions:

1. The function $\alpha(t)$ maps the contour L onto itself with reversal of the direction of traversal on it.
2. The Carleman conditions ([7]) are satisfied:

$$\alpha[\alpha(t)] = t, \tag{2}$$

$$\frac{b(t)b[\alpha(t)]}{a(t)a[\alpha(t)]} = 1. \tag{3}$$

Using the Sokhotski formulas for the limiting values of the Cauchy-type integral

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{f(\tau)}{\tau - z}\, d\tau,$$

where $f(t)$ is the unknown function, we reduce the integral equation (1) to the following boundary-value problem for the piecewise-analytic function $\Phi(z)$:

$$\Phi^+(t) + A(t)\Phi^+[\alpha(t)] - \Phi^-(t) + A(t)\Phi^-[\alpha(t)] = H(t) \quad \text{on } L, \tag{4}$$

where $A(t) = b(t)/a(t)$, $H(t) = h(t)/a(t)$, and condition (3) takes the form

$$A(t)A[\alpha(t)] = 1. \tag{3'}$$

To each solution of problem (4) vanishing at infinity there corresponds, by the formula $\Phi^+(t) - \Phi^-(t) = f(t)$, a definite solution of equation (1).

Using conditions (2) and (3′), we reduce the boundary-value problem (4) to the equivalent pair of Riemann–Carleman boundary-value problems:

$$\Phi^+[\alpha(t)] = -A[\alpha(t)]\Phi^+(t) + \frac{1}{2}\{A[\alpha(t)]H(t) + H[\alpha(t)]\} \quad \text{on } L; \tag{5}$$

$$\Phi^-[\alpha(t)] = A[\alpha(t)]\Phi^-(t) + \frac{1}{2}\{A[\alpha(t)]H(t) - H[\alpha(t)]\} \quad \text{on } L. \tag{6}$$

The Riemann–Carleman problem for the interior domain was considered by D. A. Kveselava ([8]). The solution of this problem in ([8]) is based on the study of a Fredholm integral equation whose kernel turns out to have no eigenfunctions. The same method can be applied in studying the Riemann–Carleman problem for the exterior (infinite) domain D^-. Here we arrive at a homogeneous integral equation having nontrivial solutions. An analogous relation exists between investigations, by the method of integral equations, of the solutions of the interior and exterior Dirichlet problems ([9]).

It is not difficult to verify that condition (3′) is necessary for the solvability of problems (5) and (6) for an arbitrary right-hand side $H(t)$. Let us also note that the shift $\alpha(t)$ has on L two fixed points t_0' and t_0''. Denote $\operatorname{Ind} A(t) = \operatorname{Ind} a(t) - \operatorname{Ind} b(t) = \varkappa$. We shall call the number $\varkappa$ the index of the boundary-value problem (4) and of the integral equation (1).

Theorem 1. *The homogeneous boundary-value problem (4) is solvable for any index* $\varkappa$ ($\varkappa = 2\varkappa'$ *or* $\varkappa = 2\varkappa' - 1,\ \varkappa' = 0,\ \pm 1,\ \pm 2, \dots,\ \pm p$).

For $\varkappa > 0$ *the homogeneous problem (4) has* $\varkappa'$ *linearly independent solutions if* $A(t_0') = A(t_0'') = -\lambda = 1$, *and* $\varkappa' + 1$ *solutions in the remaining cases:* $A(t_0') = A(t_0'') = -\lambda = -1$ *and* $A(t_0') = -A(t_0'') = \pm 1$.

The general solution of the problem is given by the formulas

$$\Phi^+(z) = X^+(z)\left\{R_{\varkappa'}(z) + \frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{\tau - z}\,d\tau\right\},$$

$$\Phi^-(z) = 0, \tag{7}$$

where $R_{\varkappa'}(z)$ *is a rational function with arbitrary coefficients, with a pole at the point* $z = 0$ *of order not exceeding* $\varkappa'$; $\varphi(t)$ *is a solution of the Fredholm integral equation*

$$K_+\varphi \equiv \varphi(t) + \frac{1}{2\pi i}\int_L \left[\frac{1}{\tau - t} - \frac{\alpha'(\tau)}{\alpha(\tau) - \alpha(t)}\right]\varphi(\tau)\,d\tau = \lambda R_{\varkappa'}[\alpha(t)] - R_{\varkappa'}(t);$$

K_+ *is an operator without eigenfunctions.*

For $\varkappa < 0$ *the homogeneous problem (4) has* $-\varkappa'$ *linearly independent solutions if* $A(t_0') = A(t_0'') = \lambda = -1$, *and* $-\varkappa' + 1$ *solutions in the other two cases. The general solution is given by the formulas*

$$\Phi^+(z) = 0,$$

$$\Phi^-(z) = X^-(z)\left\{P_{\varkappa'}(z) + \frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{\tau - z}\,d\tau\right\}, \tag{8}$$

where $P_{\varkappa'}(z)$ *is an arbitrary polynomial of degree not exceeding* $-\varkappa'$; $\varphi(t)$ *is a solution of the Fredholm integral equation*

$$K_-\varphi \equiv \varphi(t) - \frac{1}{2\pi i}\int_L \left[\frac{1}{\tau - t} - \frac{\alpha'(\tau)}{\alpha(\tau) - \alpha(t)}\right]\varphi(\tau)\,d\tau = P_{\varkappa'}(t) - \lambda P_{\varkappa'}[\alpha(t)].$$

The operator K_- has one eigenfunction $\varphi(t) = 1$. $X^\pm(z)$ are the canonical functions [8,10] of problems (5) and (6), determined respectively by the conditions

$$X^\pm[\alpha(t)] = \mp\lambda A[\alpha(t)] X^\pm(t).$$

For $\chi = 0$ there exists one linearly independent solution of problem (4), represented by formulas (7), where $R_{\chi'}(z) \equiv C$, if $A(t_0') = A(t_0'') = -1$, and by formulas (8), where $P_{\chi'}(z) \equiv C$, if $A(t_0') = A(t_0'') = 1$.

Theorem 2. The nonhomogeneous problem (4) is unconditionally solvable only for $\chi = 0$. The general solution is expressed by the formulas:

$$\Phi^+(z) = X^+(z)\left\{R_{\chi'}(z) + \frac{1}{2\pi i}\int_L \frac{\varphi(\tau)\,d\tau}{\tau - z}\right\},$$

$$\Phi^-(z) = X^-(z)\left\{P_{\chi'}(z) + \frac{1}{2\pi i}\int_L \frac{\varphi^*(\tau)\,d\tau}{\tau - z}\right\};$$

$\varphi(t)$ and $\varphi^*(t)$ are solutions of the Fredholm integral equations

$$K_+\varphi = \frac{1}{2}\left\{\frac{H(t)}{X^+(t)} - \frac{\lambda H[\alpha(t)]}{X^+[\alpha(t)]}\right\} + \lambda R_{\chi'}[\alpha(t)] - R_{\chi'}(t),$$

$$K_-\varphi = \frac{1}{2}\left\{\frac{H(t)}{X^-(t)} - \frac{\lambda H[\alpha(t)]}{X^-[\alpha(t)]}\right\} + P_{\chi'}(t) - \lambda P_{\chi'}[\alpha(t)].$$

For $\chi > 0$, $P_{\chi'}(z) \equiv 0$, and the solvability conditions are required:

$$\int_L t^{k-1}\varphi^*(\tau)\,d\tau = 0, \quad k = 1, 2, \ldots, \chi' - 1,$$

to which, when $A(t_0') = A(t_0'') = -1$, the condition is added

$$\int_L \left\{\frac{H(t)}{X^-(t)} + \frac{H[\alpha(t)]}{X^-[\alpha(t)]}\right\}\psi(t)\,dt = 0;$$

$\psi(t)$ is a nontrivial solution of the equation $K_-'\psi = 0$, adjoint to the equation $K_-\varphi^* = 0$. For $\chi < 0$, $R_{\chi'}(z) \equiv 0$, and the conditions

$$\int_L t^{-k}\varphi(t)\,dt = 0, \quad k = 1, 2, \ldots, -\chi'$$

must be satisfied.

Imposing the condition $\Phi^-(\infty) = 0$, we obtain the following conclusions for the integral equation (1):

1. The number of linearly independent solutions of the homogeneous equation (1) is equal (for $\chi > 0$, and also for $\chi = 0$ and $A(t_0') = A(t_0'') = 1$) or is less by one (for $\chi < 0$, and also for $\chi = 0$ and $A(t_0') = A(t_0'') = 1$) than the number of linearly independent solutions of the homogeneous problem (4).

Hence the unsolvability of the homogeneous equation follows in the following three cases:

1) $\chi = 0, \quad A(t_0') = A(t_0'') = 1;$

2) $\chi = -1;$

3) $\chi = -2, \quad A(t_0') = A(t_0'') = -1.$

2. The nonhomogeneous equation (1) is unconditionally solvable and has a unique solution if $\chi = 0$ and $A(t_0') = A(t_0'') = 1$. In the remaining cases, for solvability of this equation it is necessary and sufficient that the conditions

$$\int_L h_k(t)h(t)\,dt = 0, \quad k = 1, 2, \dots, q(\chi'),$$

be satisfied, where $h_k(t)$ are completely determined linearly independent functions not depending on $h(t)$.

Studying the singular equation adjoint to equation (1), we arrive at the following conclusions:

Theorem 3. *The indices of the adjoint and the given integral equations are related by the relation* $\chi^* = \chi - \chi_{\alpha'}$, *where* $\chi_{\alpha'} = \operatorname{Ind} \alpha'(t) = -2$.

Theorem 4. *The given homogeneous integral equation and the homogeneous equation adjoint to it have the same number of solutions.*

Thus, the theory of the singular integral equation (1) represents a distinctive interweaving of Noether theory and Fredholm theory.

I express my gratitude to Prof. F. D. Gakhov for his constant interest in, and attention to, this work.

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Received
27 V 1960

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Note: Figure translations are in progress. See original paper for figures.