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Abstract

Full Text

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ON THE EMBEDDABILITY OF SEMIGROUPS IN GROUPS

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In the paper ([5]) an example was constructed of a finitely presented (f.p.) semigroup with two-sided cancellation which is not isomorphic to any finitely presented associative system (i.e., a semigroup without the cancellation law). In the present note we shall use the terminology and notation of ([5]). In the first part of the note it is proved that every f.p. semigroup with two-sided cancellation is embeddable in a f.p. associative system. An embedding algorithm is given.

In the second part of the present note a certain class of associative systems is singled out which are isomorphic to the corresponding semigroups with two-sided (one-sided) cancellation with the same defining relations, and it is proved that these semigroups are embeddable in groups with the same defining relations.* As a consequence it is obtained that every semigroup with two-sided cancellation with one defining relation is embeddable in a group with the same relation. Hence, on the basis of Magnus' result ([3]) on the solvability of the identity problem in a group with one defining relation, it follows that in any semigroup Π_2 with one defining relation the identity problem is solvable. Consequences are also given on the solvability of the identity problem in certain semigroups Π, Π_l, and Π_{pr} with one defining relation. An example is indicated of a semigroup Π_2 with two generators and two defining relations which is not embeddable in any group.

1. Let the semigroup with cancellation Π_2 be given by the generators

$$a_1, a_2, \dots, a_n \tag{1}$$

and by the defining relations

$$A_k = B_k \quad (k = 1, 2, \dots, m). \tag{2}$$

By Π we shall denote the associative system given by the generators

$$b_1, b_2, \dots, b_n,\ c_1, c_2, \dots, c_n,\ d_1, d_2, \dots, d_n,$$

$$p_1, p_2, \dots, p_n,\ q_1, q_2, \dots, q_n,\ r_1, r_2, \dots, r_n,\ a_1, a_2, \dots, a_n$$

and by the system of defining relations

$$A_k = B_k \quad (k = 1, 2, \dots, m),$$

$$a_i b_i = 1, \qquad c_i a_i p_i q_i = 1, \qquad q_i a_j = a_j q_i,$$

$$q_i b_i = r_i d_i, \qquad r_i a_j = a_j r_i, \qquad a_i p_i r_i = r_i p_i a_i,$$

where the indices i, j independently run through all values from 1 to n. Words composed of the generators (1) will be called words of the basic alphabet of the system Π.

Lemma 1. *Let X, Y be words of the basic alphabet of the system Π. In order that the equality $X = Y$ hold in Π_2, it is necessary and sufficient that this same equality hold in the system Π.*

The proof of Lemma 1, as well as the proofs of many other assertions of this note, is omitted because of its bulk.

* The question of the embeddability of associative systems with given defining relations in a group with the same relations was formulated by A. I. Mal' cev in ([2]). There also, from this point of view, semigroups were investigated all of whose defining relations have the form $uv = xy$, where u, v, x, y are generators.

From Lemma 1 it follows

Theorem 1. *The semigroup generated in the system Π by the generators (1) is isomorphic to the semigroup Π_2.*

A **finitely generated associative system** (**a semigroup with cancellation**) is any associative system that is isomorphic to some subsemigroup of some finitely generated associative system (finitely generated semigroup with cancellation). The following theorem is a reformulation of Theorem 1.

Theorem 2. *Every finitely generated semigroup with cancellation is isomorphic to some finitely generated associative system.*

It follows from Theorem 1 that every finitely generated semigroup with cancellation can be embedded in a finitely generated associative system with two generators. This may be of interest in connection with the example constructed by Evans[4] of a finitely generated semigroup with cancellation that cannot be embedded in any semigroup with cancellation with two generators.

II. Let an associative system Π be given by generators (1) and defining relations (2), where none of the words A_k, B_k is empty. To each defining relation $A_k = B_k$ of the system (2) we assign an unordered pair of generators (a_{i_k}, a_{j_k}) such that A_k begins with a_{i_k}, and B_k with a_{j_k}, or conversely. The pair (a_{i_k}, a_{j_k}) will be called the **left pair** of the relation $A_k = B_k$. Analogously, the **right pair** corresponding to the relation $A_k = B_k$ will be the pair (a_{t_k}, a_{s_k}) such that A_k ends in a_{t_k}, and B_k in a_{s_k}, or conversely. The **left graph** of the system of relations (2) is the graph defined by all the generators occurring in the relations (2), and by all left pairs of these relations (([6], p. 39). The **right graph** of the system of relations (2) is the graph defined by the same elements and by all right pairs of the defining relations (2). We shall say that the system of relations (2) **has no left (right) cycles** if the corresponding left (right) graph contains no cycles. In doing so we assume that all words A_k and B_k are nonempty. If the system of relations (2) has neither right nor left cycles, then we shall simply say that it **has no cycles**. Notice that a system consisting of a single defining relation $A = B$ with nonempty A and B will have a left (right) cycle if and only if this relation is left- (right-) cancellable.

Theorem 3. *If the system of defining relations (2) of the associative system* Π *has no left (right) cycles, then in the system* Π *the rule of left-hand (right-hand) cancellation holds.*

Let Π_2 be a semigroup with two-sided cancellation given by generators (1) and defining relations (2), and let Γ be the group with positive alphabet (1) and the system of nontrivial defining relations (2).

Theorem 4. *If the system of relations (2) has no cycles, then the system* Π *is isomorphic to the semigroup* Π_2 *(to the semigroup* Π_l *and to the semigroup* Π_r *with the same relations).*

Lemma 2. *Let the system of relations (2) have no cycles. Let* X, Y *be words in the alphabet (1). In order that the equality* $X = Y$ *hold in* Γ*, it is necessary and sufficient that this equality hold in* Π*.*

From Lemma 2 we obtain the theorem on embeddability of finitely generated associative systems in groups.

Theorem 5. *If the system of relations (2) of the associative system* Π *has no cycles, then the system* Π *is isomorphic to a subsemigroup of the group* Γ *generated by the generators (1).*

From Theorems 3 and 4 follows the theorem on embeddability of semigroups with cancellation in groups.

Theorem 6. If the system of defining relations (2) of a semigroup with two-sided (left-sided or right-sided) cancellation Π_2 (Π_l or Π_{pr}) has no cycles, then the semigroup Π_2 (Π_l, Π_{pr}) is isomorphic to a subsemigroup of the group Γ generated by the generators (1).

Let us note some consequences of these theorems pertaining to semigroups with one defining relation.

Corollary 1. If the associative system Π is given by one defining relation $A = B$, which is irreducible both on the left and on the right, and the words A and B are both nonempty, then the system Π is isomorphic to the semigroups Π_l, Π_{pr}, and Π_2 with the same defining relation, and all of them are isomorphic to the subsemigroup generated by the words of the positive alphabet in the group Γ with the same defining relation (3).

Corollary 2. If a semigroup Π $(\Pi_l, \Pi_{\text{pr}}, \Pi_2)$ is given by one defining relation (3), which is irreducible both on the right and on the left, and the words A and B are nonempty, then the word problem is solvable in this semigroup.

The last assertion follows on the basis of the well-known result of Magnus ([3]) on the solvability of the word problem in a group with one defining relation.

The following assertion is proved very simply and, apparently, is known.

Theorem 7. If the defining relations of a c.o. semigroup Π_l (Π_{pr} or Π_2), having the form $C_i = 1$, contain all generators of this semigroup, then the semigroup Π_l (respectively Π_{pr} or Π_2) is a group.

Indeed, it is easy to see that then each generating element will have in the semigroup under consideration a left inverse and a right inverse, and the latter will be equal to one another.

Using Theorem 7, we obtain:

Corollary 3. If a semigroup Π_l (Π_{pr}) is given by one defining relation (3), which is irreducible on the right (left), then it is embeddable in the group with the same defining relation, and, consequently, the word problem is solvable in it.

Corollary 4. Every semigroup with two-sided cancellation Π_2 with one defining relation is embeddable in the group with the same defining relation.

Corollary 5. In every semigroup Π_2 with one defining relation the word problem is solvable.

In connection with Corollary 4 we note that the semigroup with two-sided cancellation Π_2, defined by generators a, b, c, d, e and defining relations $ab = cd$, $aed = ced$, is not embeddable in a group*.

Thus the question has been clarified of the minimal number of defining relations of a semigroup with two-sided cancellation Π_2 for which this semigroup cannot be embedded in a group. This number is 2.

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REFERENCES

1. A. Malcev, *Math. Ann.*, **113**, 5, 686 (1936).

2. A. Mal' tsev, *Matem. sborn.*, **8** (50), No. 2 (1940).

3. W. Magnus, *UMN*, **8**, 365 (1941).

4. T. Evans, *Proc. Am. Math. Soc.*, **3**, No. 4 (1952).

5. S. I. Adian, *DAN*, **113**, No. 6 (1957).

6. S. V. Yablonskii, *Tr. Matem. inst. im. V. A. Steklova AN SSSR*, **51**, 5 (1958).

* This example is a simplification of another example with relations $p_1p_2 = ap_1ap_2a,\ p_1p_2 = a^2p_1a^2p_2a^2$, which was pointed out to the author by P. S. Novikov. The example first constructed by A. I. Mal' tsev in the work ([1]) contains three defining relations $ax = by,\ au = bv,\ cx = dy$.

Note: Figure translations are in progress. See original paper for figures.