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Abstract

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On Topological Methods in the Theory of Functions of a Complex Variable

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§ 1.

In recent decades a new theory has arisen, relying on topological methods and making it possible to determine the number of critical points of harmonic, pseudoharmonic, analytic, and pseudoanalytic functions from prescribed singularities in a domain and from contour values. A summary exposition of this theory is given in the monograph of M. Morse ([1]). For M. Morse, the admissible singularities for harmonic functions are logarithmic poles, and for analytic functions, poles. The simultaneous presence of a polar and a logarithmic singularity at one and the same point is not allowed. Only finite domains are investigated.

In the work of F. D. Gakhov and Yu. M. Krikunov ([2]), the results of M. Morse' s monograph are generalized and refined in various directions. Some results of the work of F. D. Gakhov and Yu. M. Krikunov have been generalized by T. A. Kolomiytseva ([3]). As in the monograph ([1]), so also in the works ([2], [3]), only single-valued harmonic functions are studied.

In the present note we study the case when a function $f(z)$, analytic in a given domain G and continuously extendable to the boundary G, has a finite number of interior and boundary singular points a_k. In a neighborhood of a_k, the function $f(z)$ has the form

$$f(z) = (z - a_k)^{p_k} \left[g_k(z) \ln^{q_k}(z - a_k) + \psi_k(z)\right] + C, \tag{1}$$

if $z = a_k$ is a finite point;

$$f(z) = z^{p_k} \left[g_k(z) \ln^{q_k} z + \psi_k(z)\right] + C', \tag{2}$$

if $z = a_k$ is the point at infinity; p_k and q_k are arbitrary integers, $q_k \neq 0$; the functions $g_k(z)$ and $\psi_k(z)$ are analytic in a neighborhood of a_k, if a_k is an interior point of the domain G, and continuous at a_k and have first derivatives on the boundary G in a neighborhood of a_k, if a_k is a boundary point; $|g_k(a_k)| + |\psi_k(a_k)| \neq 0$; C is a complex constant; G is a finite or infinite domain bounded by α Jordan curves $(\Gamma_1, \Gamma_2, \dots, \Gamma_\alpha) = \Gamma$. Since an infinite domain G can be mapped one-to-one and with preservation of orientation onto a finite domain D, in the investigation it is sufficient to consider the case of a finite domain G.

Definition. A point a_k in a neighborhood of which $f(z)$ has a representation of the form (1) or (2) will be called a **power-logarithmic point** of the function $f(z)$. In the special case when $g_k(z) \equiv 0$ and $p_k = 1$, a_k is an ordinary point of the function $f(z)$. The number p_k corresponding to the point a_k will be called the **order of the power-logarithmic point** a_k.

We shall call a boundary curve Γ_k **exterior** if any of its points can be connected with the infinitely distant point by a Jordan curve lying entirely outside G. If such a connection is impossible, then the boundary curve Γ_k will be called **interior**.

In the cases considered here the function $u(x, y) = \operatorname{Re} f(z)$ has in G a finite number of discontinuity lines leading from the points a_k to the exterior boundary curve Γ_α. The discontinuity lines of $u(x, y)$ are cuts in G that single out single-valued branches of logarithms.

Let L_k be a cut in G connecting the point a_k with the exterior boundary curve Γ_α, and let t be the complex coordinate of the points of the line L_k. By the function $\ln(z - a_k)$ we shall mean any of its branches, single-valued in the domain G cut along L_k, and taking on the left side of L_k the value $\ln(t - a_k)$. If $q_k \neq 1$, then we shall assume that the cut L_k, issuing from the point $a_k = \alpha_k + i\beta_k$, passes along the line $t = x + i\beta_k,\ x \leq \alpha_k$, and that the function $g_k(z)$ satisfies the condition $\operatorname{Im} g_k(x + i\beta_k) = 0$ on L_k. By elementary calculations one can show that in this case

$$u^+(x, y) - u^-(x, y) = 0 \quad \text{on } L_k.$$

If $q_k = 1$, then, putting in (1)

$$w_k(z) = u_k + iv_k = (z - a_k)^{p_k} g_k(z),$$

we obtain on L_k

$$u^+(x, y) - u^-(x, y) = 2\pi v_k.$$

In this case we shall consider admissible only those cuts L_k on which the function v_k is constant, or piecewise constant. It can be shown that, when the function $w_k(z)$ is single-valued in G, there always exists an admissible cut L_k connecting a_k with the exterior boundary curve Γ_α. If $w_k(z)$ is multivalued in G, then we shall assume that only those cases are considered for which admissible cuts exist.

The admissible cuts L_k can always be replaced by admissible cuts C_k so that the following conditions are satisfied:

1) The cuts C_k do not intersect and have no common points with the interior boundary curves; they do not pass through interior critical points of $u(x, y)$ and end on Γ at ordinary (noncritical) points of the function $u(x, y)$.

2) The functions u^+ and u^- (the limiting values of the function $u(x, y)$ on C_k) have no more than a finite number of points of relative extremum on C_k; they increase when approaching the point of intersection of C_k with the exterior boundary curve Γ_α.

Assume first that on the boundary Γ there are no power-logarithmic points and that the function $u(x, y)$ satisfies the boundary conditions A or C (([1]), p. 60) everywhere on Γ, except, possibly, at the points of intersection of the cuts C_k with Γ_α. Since the points of intersection of C_k with Γ_α are ordinary points of the function $u(x, y)$, the contribution of the boundary index of the function $u(x, y)$ from Γ_α is computed according to the rules set forth in the monograph ([1]). By the boundary index of the function $u(x, y)$ along the contour Γ relative to G we shall mean the sum of the contributions from each curve Γ_k.

Theorem 1. *Let I be the boundary index of the function $u(x, y) = \operatorname{Re} f(z)$ along the contour Γ relative to G. Then the equality*

$$\sum_{k=1}^{m} (1 - p_k) = 2 - \alpha + I, \tag{3}$$

holds, where m is the number of power-logarithmic points of $f(z)$ in G; p_k are the orders of these points; α is the number of boundary curves.

In the proof of the theorem the decisive role is played by the following

Lemma. Let $z = a_k$ be a power-logarithmic point of $f(z)$ of order p_k, and let L_k be an admissible cut issuing from a_k, on which $u^+(x, y) - u^-(x, y) = 0$. There exists a sufficiently small number r_0 such that the increment I_{γ_k} of the boundary index of the function $u(x, y)$ from the circle γ_k ($|z - a_k| = r_0$), relative to the domain $|z - a_k| > r_0$, is equal to p_k.

Denote by G_0 the domain obtained from G by removing the closed circular neighborhoods of the points a_k, not containing other critical points except a_k, and all cuts on which

$$u^+(x, y) - u^-(x, y) = 2\pi c \neq 0; \tag{4}$$

denote by Γ_0 the boundary of G_0. It follows from (4) that on the admissible cut C_k the functions $u^+(x, y)$ and $u^-(x, y)$ have the same number of relative extremum points; moreover, the extremum points of $u^+(x, y)$ on the left bank coincide with the extremum points of $u^-(x, y)$ on the right bank of the cut. Since an extremum entering on the left bank of the cut is exiting on the right, it is easy to show that the increment of the boundary index of $u(x, y)$ from C_k

relative to G_0 is equal to -1. With the aid of the lemma and the last conclusion, it is not difficult to obtain a proof of the theorem.

§ 2. Let $a_k = z(s_k)$ be an arbitrary point of the contour Γ, ordinary, angular, or a return point, and suppose that in a neighborhood of a_k the contour Γ is smooth to the left and to the right of a_k, except, perhaps, for the point a_k. Suppose that $\Omega(s) = u(s) + iv(s)$ is the boundary value of an analytic function $f(z)$, having in $\overline{G}$ a finite number of interior and boundary power-logarithmic points a_k of orders p_k, and that $u(s)$ between the points a_k satisfies boundary conditions A or C. By elementary operations, taking into account the equation of the contour Γ, the function $\Omega(s)$ in a neighborhood of a_k is transformed to the form

$$\Omega(s) = [z(s) - a_k]^{p_k}\{\Phi(s)\ln^{q_k}[z(s) - a_k] + \Psi(s)\} + C,$$

where $|\Phi(s_k)| + |\Psi(s_k)| \neq 0$.

Around each boundary point a_k describe a circle γ_k ($|z - a_k| = r$) of so small a radius r that it intersects Γ only at two points b_{1k} and b_{2k}, situated at a positive distance from the points of relative extremum of $u(s)$ on Γ, so that the function $u(s)$ on the arcs $b_{1k}a_k$ and $a_{kb_{2k}}$ varies strictly monotonically, and so that the intersection of $\overline{G}$ with the disk $|z - a_k| \leq r$ contains no other power-logarithmic points of the function $f(z)$, except the point a_k. Denote by G_0 the domain obtained from G by removing the closed neighborhoods of the boundary points a_k cut off by the circles γ_k; denote by Γ_0 the boundary of G_0; denote by δ_k the closed arc of the circle γ_k entering into Γ_0; $\Gamma_* = \Gamma_0 - \sum \delta_k$. Let $U(x, y)$ be the function defined by the values $\operatorname{Re} f(z)$ in $\overline{G}_0$.

By the **boundary index** of the function $u(s)$ along the contour Γ relative to G we shall mean the boundary index of the function U along Γ_0 relative to G_0.

On Γ_*, $U = u(s)$. The boundary values of U are unknown only on the arcs δ_k. However, as the following theorem shows, the increment of the boundary index of the function U from the arc δ_k is determined uniquely.

Denote

$$t = (-1)^{q_k}\operatorname{Re}\big[(z - a_k)^{p_k}\Phi(s_k)\big], \qquad \text{if } q_k > 0 \text{ or } \Psi(s_k) = 0;$$

$$t = \operatorname{Re}\big[(z - a_k)^{p_k}\Psi(s_k)\big], \qquad \text{if } q_k < 0 \text{ or } \Phi(s_k) = 0.$$

Theorem 2. Let I_t be the increment of the boundary index of the function t from the closed arc δ_k relative to G_0; let I_{a_k} be the increment of the boundary index of the function U from the closed arc δ_k relative to G_0. If on the arcs $b_{1k}a_k$ and $a_{kb_{2k}}$ the function t is strictly monotone, then $I_t = I_{a_k}$.

It follows from what has been said that relation (3) is also valid in the case when $f(z)$ has a finite number of boundary power-logarithmic points a_k, if I is understood there as the boundary index of the function $u(s)$, defined in § 2.

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REFERENCES

1. M. Morse, *Topological methods in the theory of functions of a complex variable*, IL, 1951.

2. F. D. Gakhov, Yu. M. Krikunov, Izv. Akad. Nauk SSSR, Ser. Mat., 20, 207 (1956).

3. T. A. Kolomiitseva, Izv. Vyssh. Uchebn. Zaved., Mathematics, No. 3 (10), 91 (1959).

Note: Figure translations are in progress. See original paper for figures.