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Abstract

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ON A GENERAL REPRESENTATION OF SOLUTIONS OF THE AXISYMMETRIC STATIONARY PROBLEM

(Presented by Academician I. N. Vekua on 1 II 1960)

The present note is devoted to the study of general properties of solutions of the system of differential equations

$$\frac{\partial}{\partial x}(rV_x) + \frac{\partial}{\partial r}(rV_r) = 0,$$

$$\frac{\partial}{\partial x}V_r - \frac{\partial}{\partial r}V_x = 0, \tag{1}$$

which express the incompressibility of the fluid and the absence of sources and vortices in the field of an axisymmetric flow in a cylindrical coordinate system.

1. We introduce the complex notation for system (1). Let $z = x + ir$, $f = V_r + iV_x$, $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial r}\right)$; then system (1) is equivalent to a single complex equation

$$\frac{\partial f}{\partial z} - \frac{1}{4ir}f - \frac{1}{4ir}\overline{f} = 0. \tag{1'}$$

The coefficients of this equation on the x-axis become infinite, exactly of first order; therefore the well-developed theory of generalized analytic functions ([1]) is not applicable here. However, owing to their simple structure and to their analyticity with respect to r for $r \neq 0$, we can use another method of I. N. Vekua, consisting in continuing solutions into the domain of complex arguments ([2,3]). From the ellipticity of system (1) for $r \neq 0$ it follows that every generalized solution $f(z) \equiv f(x, r)$ of equation $(1')$ will be analytic in a neighborhood of any point (x_0, r_0), $r_0 \neq 0$, so that, continuing $f(x, r)$ into the domain of complex values of x and r, and putting

$$z = x + ir, \quad \xi = x - ir,$$

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we obtain the function

$$F(z,\xi) = f\left(\frac{z+\xi}{2}, \frac{z-\xi}{2i}\right),$$

analytic in z, ξ and coinciding with $f(x,r)$ on the plane $z = \xi$. This function, as is easy to show ([2]), satisfies the equation

$$\frac{\partial F(z,\xi)}{\partial \xi} - \frac{1}{2}\frac{1}{z-\xi}F(z,\xi) - \frac{1}{2}\frac{1}{z-\xi}F^*(\xi,z) = 0, \tag{2}$$

where the asterisk denotes the function constructed from $F(z,\xi)$ according to the formula $F^*(\xi,z) = \overline{F(\bar{\xi},\bar{z})}$. A general representation of solutions of equations of the form (2) is given in ([2]). Our aim is to express effectively the resolvents arising here, using the simplicity of the coefficients of equation (2) and the close connection between equation (2) and the classical Euler–Poisson equation.

2. Let us consider the general equation of the form (2):

$$\frac{\partial F(z,\zeta)}{\partial \zeta} + A(z,\zeta)F(z,\zeta) + B(z,\zeta)F^*(\zeta,z) = 0 \tag{2'}$$

in some bicylindrical domain $(D_z \times D_\zeta)$, $D_\zeta = D_{\bar{z}}$, in which the coefficients are analytic and regular. It is easy to show that if the coefficients A and B satisfy the identities

$$\frac{\partial A(z,\zeta)}{\partial z} - B(z,\zeta)B^*(\zeta,z) = -C_1(z,\zeta)A(z,\zeta),$$

$$\frac{\partial B(z,\zeta)}{\partial z} - B(z,\zeta)A^*(\zeta,z) = -C_1(z,\zeta)B(z,\zeta), \tag{3}$$

where $C_1(z,\zeta)$ is a certain analytic and regular function in $(D_z \times D_\zeta)$, then any solution of equation $(2')$ also satisfies the second-order equation

$$\frac{\partial^2 F(z,\zeta)}{\partial z\,\partial \zeta} + A(z,\zeta)\frac{\partial F(z,\zeta)}{\partial z} + C_1(z,\zeta)\frac{\partial F(z,\zeta)}{\partial \zeta} = 0. \tag{4}$$

The converse assertion is not valid. Indeed, the general representation of the solutions of equation (4) has the form ((3), Ch. I)

$$F(z,\zeta) = \alpha G(z_0,\zeta_0,z,\zeta) + \int_{z_0}^{z} \varphi(t)G(t,\zeta_0,z,\zeta)\,dt + \int_{\zeta_0}^{\zeta} \varphi_1(\tau)G(z_0,\tau,z,\zeta)\,d\tau, \tag{5}$$

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where $\varphi(t)$ and $\varphi_1(\tau)$ are arbitrary analytic functions in the domains D_z, D_ζ, respectively, $z_0 \in D_z$, $\zeta_0 \in D_\zeta$, $\alpha = \text{const}$, and $G(z, \zeta, t, \tau)$ is the Riemann function of equation (4), and thus contains two analytic functions; meanwhile, the general representation of the solutions of equation (2′) contains one analytic function. Therefore, in order that the solutions of equation (4) also satisfy equation (2′), there must exist some dependence between $\varphi(t)$ and $\varphi_1(\tau)$. This dependence is clarified by the following theorem.

Theorem 1. *Any solution of equation (2′), under assumption (3), can be represented by formula (5), in which G is the Riemann function of equation (4), z_0 is a fixed point of D_z, $\zeta_0 = \bar{z}_0$, α is an arbitrary constant, and the functions $\varphi(t)$, $\varphi_1(\tau)$ are related by the ordinary differential equation*

$$\varphi(z) = -\frac{1}{B^*(\zeta_0, z)} \frac{d\varphi_1^*(z)}{dz} + \frac{1}{B^{*2}(\zeta_0, z)} \left[\frac{\partial B^*(\zeta_0, z)}{\partial z} - C_1(z, \zeta_0) B^*(\zeta_0, z) \right] \varphi_1^*(z), \tag{6}$$

where $\varphi_1(\zeta_0) = -\bar{\alpha} B(z_0, \zeta_0)$ and, of course, $B^*(\zeta_0, z) \neq 0$.

Conversely, if the functions $\varphi(t)$ and $\varphi_1(\tau)$ are related by equation (6), are regular in D_z and D_ζ, respectively, and $\varphi_1(\zeta_0) = -\bar{\alpha} B(z_0, \zeta_0)$, then formula (5) represents a solution of equation (2′) for any value of the constant α.

3. The general representation of all regular solutions of equation (2′) is now obtained from formula (5) by eliminating the function $\varphi(t)$ from equation (6) and by termwise integration:

$$F(z, \zeta) = -\frac{\varphi_1^*(z)}{B^*(\zeta_0, z)} G(z, \zeta_0, z, \zeta) + \int_{\zeta_0}^{\zeta} G(z_0, \tau, z, \zeta) \varphi_1(\tau)\, d\tau +$$
$$+ \int_{z_0}^{z} \left[\frac{\partial G(t, \zeta_0, z, \zeta)}{\partial t} - C_1(t, \zeta_0) G(t, \zeta_0, z, \zeta) \right] \frac{\varphi_1^*(t)}{B^*(\zeta_0, t)}\, dt. \tag{7}$$

The function $\varphi_1(\tau)$ is chosen in this formula quite arbitrarily, with only regularity preserved, since the condition $\varphi_1(\zeta_0) = -\bar{\alpha} B(z_0, \zeta_0)$, by virtue of the arbitrariness of α, imposes no restriction on φ_1. Therefore the above-mentioned resolvents Γ_1, Γ_2 from paper (2), taking into account certain preliminary transformations carried out there, are expressed by the formulas

$$\Gamma_1(z, \zeta, t, \zeta_0) = -\frac{1}{G(z, \zeta_0, z, \zeta)} \left[\frac{\partial G(t, \zeta_0, z, \zeta)}{\partial t} - C_1(t, \zeta_0)\, G(t, \zeta_0, z, \zeta) \right],$$

$$\Gamma_2(z, \zeta, z_0, \tau) = -\frac{1}{G(z, \zeta_0, z, \zeta)} G(z_0, \tau, z, \zeta)\, B(z_0, \tau). \tag{8}$$

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4. The conditions (3) for equation (2), as is not difficult to verify, are satisfied for the value of the function

$$C_1(z,\zeta) = \frac{3}{2}\,\frac{1}{z-\zeta}.$$

Therefore equation (4) in our case is an Euler–Poisson equation with values $\beta' = 1/2,\ \beta = 3/2$. The Riemann function is then expressed through the Gauss hypergeometric function (see, for example, ([4])). For our values of the parameters β' and β, it can even be expressed through the second complete elliptic integral $E(z)$; as simple transformations show, the Riemann function has the form

$$G(z,\zeta,t,\tau) = \frac{2}{\pi}\,\frac{\zeta - z}{\tau - t}\sqrt{\frac{\tau - z}{\zeta - t}}\,E(\sqrt{\sigma}), \qquad \sigma = \frac{(z-t)(\zeta-\tau)}{(z-\tau)(\zeta-t)}.$$

Equation (6) in this case has the form

$$\varphi(z) = 2(\zeta - z)\frac{d\varphi_1^*(z)}{dz} - 5\varphi_1^*(z).$$

As the domain D_z one may take any domain in the upper half-plane. Then the general representation of equation (2) can be written in the form

$$\begin{aligned} F(z,\zeta) = \frac{2}{\pi}\,\frac{1}{\zeta - z}\Bigg\{ & \alpha(\zeta_0 - z_0)\sqrt{\frac{\zeta - z_0}{\zeta_0 - z}}\,E\left(\sqrt{\frac{(z_0 - z)(\zeta_0 - \zeta)}{(z_0-\zeta)(\zeta_0 - z)}}\right) \\ & + \int_{z_0}^{z}\varphi(t)(\zeta_0 - t)\sqrt{\frac{t-\zeta}{z-\zeta_0}}\,E\left(\sqrt{\frac{(t-z)(\zeta_0-\zeta)}{(t-\zeta)(\zeta_0 - z)}}\right)dt \\ & + \int_{\zeta_0}^{\zeta}\varphi_1(\tau)(\tau - z_0)\sqrt{\frac{\zeta - z_0}{\tau - z}}\,E\left(\sqrt{\frac{(z_0 - z)(\tau-\zeta)}{(z_0-\zeta)(\tau - z)}}\right)d\tau\Bigg\}. \end{aligned} \tag{9}$$

From the results of ([3]), Chap. I, it follows that the formula $f(x,r) = F(z,\bar{z})$ exhausts all regular solutions of equation (1′).

5. The representation (9) is considerably simplified for solutions possessing first derivatives bounded up to the axis of symmetry $r = 0$. Namely, suppose that a segment L of the axis $r = 0$ is part of the boundary of the domain D_z. Let $f(x,r)$ be bounded at the point $x_0 \in L$ as the point z_0 approaches x_0 along a path not tangent to L. Then formula (9) can be written in the form

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$$F(z,\zeta) = \frac{2}{\pi}\frac{1}{\zeta - z}\left\{ \int_{x_0}^{z} \Phi(t)\sqrt{\frac{t-\zeta}{z-x_0}}\, E\left(\sqrt{\frac{(t-z)(x_0-\zeta)}{(t-\zeta)(x_0-z)}}\right) dt + \int_{x_0}^{\zeta} \Phi_1(\tau)\sqrt{\frac{\zeta-x_0}{\tau-z}}\, E\left(\sqrt{\frac{(\tau-\zeta)(x_0-z)}{(\tau-z)(x_0-\zeta)}}\right) d\tau \right\}, \tag{10}$$

where $\Phi_1(\tau) = \varphi_1(\tau)(\tau - x_0)$, $\Phi(t) = \varphi(t)(x_0 - z)$, so that the equation

$$\Phi(z) = 2(x_0 - z)\frac{d\Phi_1^*(z)}{dz} - 3\Phi_1^*(z). \tag{11}$$

is satisfied.

The necessary and sufficient conditions for boundedness of the solution $F(z,\zeta)$ for $z = \zeta$, i.e., the conditions for boundedness of the solution $f(x,r) = F(z,\bar{z})$ on the segment L, are established by the following theorem.

Theorem 2. Every solution f of equation $(1')$ that is regular on the segment L is representable by the formula $f(x,r) = F(z,\bar{z})$, where $F(z,\zeta)$ is expressed by formula (10), with the functions $\Phi(t)$, $\Phi_1(\tau)$, analytic and regular in the domains D_z, D_ζ, respectively, satisfying the condition $\operatorname{Re}\Phi = \operatorname{Re}\Phi_1 = 0$ on L, i.e., admitting analytic continuation by the principle of symmetry to the whole domain $\Delta = D_z + L + D_\zeta$. In the entire domain of definition these functions are connected by the differential equation (11). Conversely, if the functions $\Phi(t)$ and $\Phi_1(\tau)$ are connected by equation (11) and satisfy the conditions $\operatorname{Re}\Phi = \operatorname{Re}\Phi_1 = 0$ on L, then the function $F(z,\zeta)$, constructed by formula (10), is analytic in the entire domain $\Delta_z \times \Delta_\zeta$, so that the corresponding solution $f(x,r) = F(z,\bar{z})$ of equation $(1')$ is regular on the segment L.

If it is not assumed that $\Phi(z)$ and $\Phi_1(\zeta)$ satisfy the conditions $\operatorname{Re}\Phi = \operatorname{Re}\Phi_1 = 0$ on L, then $F(z,\zeta)$ will have a pole of first order in the plane $z = \zeta$. Let us also note that it is sufficient to require the fulfillment of the conditions of the theorem only for the function Φ_1; for the function Φ these conditions then follow from equation (11).

Representation (7) together with Theorem 2 make it possible to investigate the problem of the flow, by a stationary unbounded axisymmetric flow of an incompressible fluid, past a body of revolution by the method of singular integral equations on a contour.

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