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MATHEMATICS

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THE MULTIDIMENSIONAL MOMENT PROBLEM

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In the note (1) the authors proposed a method for obtaining integral representations of positive-definite functionals on nuclear spaces. The concrete examples considered in (1) concerned spaces of functions defined on the whole k-dimensional space R^k or on a part of it. In the present note we derive, as consequences of the general scheme (1) (for details see (2)), a number of results on positive-definite functionals over spaces of sequences.

Definition 1. A continuous linear functional T on an algebra Φ with involution $\varphi \to \varphi^*$ is called **positive-definite** if $T(\varphi\varphi^*) \geqslant 0$ for all $\varphi \in \Phi$.

1. Let $\Phi = M(a_{np})$ be a countably normed space of sequences $\varphi = \{\varphi_n,\ n = 0, 1, \dots\}$ with the system of norms

$$|\varphi|_p = \sum_{n=0}^{\infty} |\varphi_n| a_{np}, \qquad p = 1, 2, \dots, \tag{1}$$

where $\{a_{np}\}$ is a countable set of sequences of positive numbers, and the following conditions are satisfied:
a) $a_{np} \leqslant a_{n,p+1}$;
b) $\sum_{n=0}^{\infty} a_{np}/a_{n,p+1} < \infty$; and
c) $a_{n+m,p} \leqslant a_{n,p+1} a_{m,p+1}$ for all n, m, p. Multiplication in M is defined by the relations

$$(\varphi\psi)_n = \sum_{0}^{n} \varphi_{n-l}\psi_l, \qquad n = 0, 1, \dots, \tag{2}$$

i.e. M may be regarded as the algebra of formal power series $\left\{\sum_0^\infty \varphi_n X^n\right\}$. Condition c) evidently ensures continuity of multiplication (2) in the topology

(1), while condition b) ensures that the topology (1) is nuclear (see (^{3})). The algebra M has a unit $e = \{1, 0, 0, ...\}$. The involution is given as complex conjugation in each coordinate: $(\varphi^*)_n = \overline{\varphi_n}$. If A is the operator of multiplication by the generator X, then the proper functional g_λ: $A' g_\lambda = \lambda g_\lambda$, normalized by the condition $g_\lambda(e) = 1$, has coordinates $g_\lambda = \{\lambda^n,\ n = 0, 1, ...\}$, i.e.

$$g_\lambda(\varphi) = \sum_0^\infty \lambda^n \varphi_n.$$

Theorem 1. *Every positive-definite functional T on M admits the representation*

$$T(\varphi) = \int_{-\infty}^{\infty} g_\lambda(\varphi)\, d\sigma(\lambda), \tag{3}$$

where $\sigma(\lambda)$ is a certain positive measure concentrated on the interval $(-\rho, \rho)$,

$$\rho = \sup_p \lim_{n\to\infty} \sqrt[n]{a_{np}}, \quad 0 \le \rho \le \infty.$$

Definition 2. A sequence s_n, $n = 0, 1, ...$ ($s_n > 0$), will be called a sequence of type K if

$$\int^\infty \frac{\log S(r)}{r^2}\, dr = \infty, \qquad \text{where} \quad S(r) = \sup_{n\ge 0} \frac{r^n}{s_n}. \tag{4}$$

Condition (4) is essential in the question of the uniqueness of the measure σ in the representation (3).

Theorem 2. *The measure σ in the representation (3) of any positive-definite functional T on M is unique if and only if all sequences $\{a_{np}\}$, $p = 1, 2, ...$, are sequences of type K.*

Sufficiency is a consequence of Carleman' s theorem (^{4,5}) on the determinate moment problem for a positive-definite sequence of type K. Necessity is proved (by contradiction) with the aid of the solution of Watson' s problem (see, for example, (^{5})). Theorem 2 is a discrete analogue of a result of E. Wulff (^{6}).

Just as the space M is defined, the space Φ of k-fold sequences (formal power series) is defined: in formulas a)—c) and (1), (2) the indices n (and m) are to be understood as vectors in R^k with nonnegative integral coordinates.

Theorem 3. *Let Φ be the space of k-fold sequences with topology (1) and multiplication (2), and suppose*

$$a_{np} \leq \prod_{j=1}^{k} b_{n_j,j,p},$$

where $b_{l,j,p}$, $l = 0, 1, \ldots$, *are sequences of type* K, *and for* $\{a_{np}\}$ *the conditions a)–c) are fulfilled. Then every positive-definite functional* T *on* Φ *admits the representation*

$$T(\varphi) = \int_{R^k} g_\lambda(\varphi)\, d\sigma(\lambda),$$

where

$$g_\lambda(\varphi) = \sum_{0}^{\infty} \varphi_{n_1,\ldots,n_k} \lambda_1^{n_1} \cdots \lambda_k^{n_k},$$

and $\sigma(\lambda)$ *is a positive uniquely determined measure on the* k*-dimensional space* R^k.

2. Definition 3. A continuous operator B in the algebra Φ will be called U-characteristic if it has a continuous inverse B^{-1} and $B\varphi \cdot \psi^* = \varphi \cdot (B^{-1}\psi)^*$ for all $\varphi, \psi \in \Phi$.

It is obvious that $B\varphi \cdot (B\psi)^* = \varphi \cdot \psi^*$, and in the Hilbert space H_T generated by any positive-definite functional T on Φ, the closure of B is a unitary operator.

Suppose that for each complex λ the space of eigenfunctionals $\Phi'_\lambda = \{\chi : B'\chi = \lambda\chi\}$ of the U-characteristic operator B is at most one-dimensional. Suppose that $\chi_\lambda(e) = 1$ for each λ (or $\chi_\lambda(\delta_n) \to 1$ on the unit sequence δ_n—see (1)). Then it can be shown that:

a) for all λ, $\chi_\lambda(\varphi\psi) = \chi_\lambda(\varphi)\chi_\lambda(\psi)$, i.e. χ_λ are multiplicative functionals; b) for all λ, $\chi_\lambda(\varphi) = \chi_{1/\bar{\lambda}}(\varphi^*)$; c) if $|\lambda| = 1$, then χ_λ is an indecomposable positive functional.

We state the Gelfand-Kostyuchenko theorem for the case of unitary operators (at once for a system of operators).

Theorem 4. *Let* Φ *be a nuclear space,* (φ, ψ) *a continuous scalar product on* Φ*, and let* H *be the completion of* Φ *with respect to this scalar product. Suppose that on* Φ *there exists a system of continuous commuting operators* B_l, $l = 1, 2, \ldots, k$, *having continuous inverses* B_l^{-1}, *and suppose that* $(B_l\varphi, \psi) = (\varphi, B_l^{-1}\psi)$ *for all* $\varphi, \psi \in \Phi$.

Then the system of common eigenfunctionals χ_λ, $\lambda = (\lambda_1, \ldots, \lambda_k)$, $B'_l\chi_\lambda = \lambda_l\chi_\lambda$, for $|\lambda_l| = 1$, $l = 1, 2, \ldots, k$, is complete in the sense that

$$(\varphi, \psi) = \sum_{\nu} \int_{S^k} \chi_\lambda^\nu(\varphi)\, \overline{\chi_\lambda^\nu(\psi)}\, d\sigma^\nu(\lambda),$$

where σ^ν are positive measures on the k-dimensional torus S^k.

If there exists no more than one common eigenfunctional χ_λ for each λ, $|\lambda_l| = 1$, then

$$(\varphi, \psi) = \int_{S^k} \chi_\lambda(\varphi)\overline{\chi_\lambda(\psi)}\, d\sigma(\lambda),$$

and the measure σ in this representation is unique.

The case considered in Theorem 4 is comparatively simple: the unitary operators are bounded in H, and from their commutativity on Φ there follows the commutativity of their spectral families $E^l_{\vartheta_l}$, $l = 1, \dots, k$, $\vartheta_l = \arg \lambda_l$.

The general scheme of the work (^{1}) and Theorem 4 make it possible to obtain Theorem 5.

Theorem 5. Let Φ be the algebra of k-fold sequences $\{\varphi_n,\ n = (n_1, \dots, n_k),\ n_l = 0, \pm 1, \pm 2, \dots\}$ with multiplication

$$(\varphi\psi)_n = \sum_{i+j=n} \varphi_i \psi_j$$

and involution $(\varphi^*)_n = \overline{\varphi_{-n}}$, with some nuclear topology in which multiplication is continuous; let Φ contain all finite sequences. Then every positive-definite functional T on Φ has a representation

$$T(\varphi) = \int_{S^k} \sum_{-\infty}^{\infty} \varphi_n e^{i(n,\vartheta)}\, d\sigma(\vartheta), \qquad (n, \vartheta) = \sum_{l=1}^{k} n_l \vartheta_l,$$

where σ is a uniquely determined positive measure on the k-dimensional torus S^k.

3. It is well known (see, for example, (^{4})) that every one-dimensional sequence T_n, $n = 0, 1, \dots$, satisfying the condition

$$\sum_{0}^{N} T_{n+m} \xi_n \overline{\xi_m} \geq 0, \tag{5}$$

where ξ_n, $n = 0, 1, \dots, N$, is any finite set of numbers, admits the representation

$$T_n = \int_{-\infty}^{\infty} \lambda^n\, d\sigma(\lambda) \tag{6}$$

with a positive measure σ. Carleman established that if $|T_n|$ is a sequence of type K, then the measure σ in representation (6) is unique.

Theorem 6. If for every positive-definite sequence T_n such that $|T_n| \leq t_n$, $n = 0, 1, ...$, the measure in representation (6) is unique, then $\{t_n\}$ is a sequence of type K.

This theorem shows that Carleman' s condition is necessary for the stability of determinacy of the moment problem. It is proved in the same way as the necessity in Theorem 2.

In the multidimensional case, i.e. $n = (n_1, \dots, n_k)$, conditions (5) are insufficient, generally speaking (7), for there to exist a positive measure σ giving the representation

$$T_n = \int_{R_k} \lambda_1^{n_1} \cdots \lambda_k^{n_k} \, d\sigma(\lambda). \tag{7}$$

But under restrictions on the growth of the sequence T_n such a representation proves to be possible; namely, the following holds:

Theorem 7. Let T_n, $n = (n_1, \dots, n_k)$, be a k-fold sequence, and suppose that a) it satisfies condition (5); b) for every m the sequence of the index n_j, $j = 1, \dots, k$,

$$Z_{n_j, j} = \sum_{\substack{n_i = 0, 2m \\ i \neq j}} T_{n_1, \dots, n_k}$$

is a sequence of type K. Then the representation (7) holds, and the measure σ is uniquely determined.

For $k = 2$ an analogous result was obtained by Devinatz (8).

Let us note some consequences of Theorem 7.

Theorem 8. Let $\sum_\Omega$ be the direct sum of lines (Ω is a set of arbitrary cardinality) and let T_n be a function on the integer nonnegative lattice $\mathfrak{A}$ in $\sum_\Omega^1$, positive-definite in the sense that

$$\sum T(n+m) \xi(n) \overline{\xi(m)} \geq 0$$

for every element $\xi = \{\xi(n)\} \in \sum_{\mathfrak{A}}$. Suppose that for every finite set $G = \{\omega_1, \dots, \omega_p\}$ in Ω the p-fold positive-definite sequence T_p, $n_\omega = 0$ for $\omega \notin G$, satisfies condition b) of Theorem 7. Then the representation

$$T_n = \int_{R^\Omega} \prod_{\omega \in \Omega} \lambda_\omega^{n_\omega} \, d\sigma(\lambda),$$

holds, where σ is a positive uniquely determined measure in the direct product R^{Ω} of lines.

This theorem follows directly from Theorem 7 and A. N. Kolmogorov' s theorem (^{9}) on the extension of a measure from cylindrical sets.

Theorem 9. Under the conditions of Theorem 7 the polynomials

$$\sum_{0}^{N} c_{n_1,\dots,n_k} \lambda_1^{n_1} \cdots \lambda_k^{n_k}$$

are dense in the space L_σ^2 of functions on R^k whose squares are summable with respect to the measure σ giving the representation (7).

An analogous proposition holds under the conditions of Theorem 8.

In conclusion we give some sufficient conditions for the solvability of the multidimensional operator moment problem.

Theorem 10. Let A_n, $n = (n_1, \dots, n_k)$, be a sequence of bounded operators in a Hilbert space H, positive-definite in the sense that

$$\left(\sum_{n,m=0}^{N} A_{n+m} \xi_n \overline{\xi_m} h, h \right) \geq 0$$

for every set of numbers ξ_n, $\eta_j = 0, 1, \dots, N$, and every vector $h \in H$. Suppose that the sequence of norms $a_n = |A_n|$ satisfies condition b) of Theorem 7. Then there exists an operator-valued positive measure $E(\lambda)$ on R^k such that

$$A_n = \int_{R^k} \lambda_1^{n_1} \cdots \lambda_k^{n_k} \, dE(\lambda),$$

and the measure $E(\lambda)$ giving this representation is unique.

It is possible (analogously to Theorem 8) to give a solution of the infinite-dimensional operator moment problem. The one-dimensional operator moment problem in a somewhat different formulation was considered in (^{10}).

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Note: Figure translations are in progress. See original paper for figures.

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