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1960

SovietRxiv

Abstract

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HYDROMECHANICS

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ON AN EXACT SOLUTION OF THE EQUATIONS OF SPATIAL UNSTEADY GAS FLOW OF THE DOUBLE-WAVE TYPE

(Presented by Academician L. I. Sedov, 17 VI 1960)

The system of equations of spatial unsteady motion of a polytropic gas has the form

$$\frac{\partial u_j}{\partial x_4} + u_k \frac{\partial u_j}{\partial x_k} + \frac{\partial u_4}{\partial x_j} = 0 \qquad \left(u_4 = \frac{a^2}{\lambda};\ \lambda = \gamma - 1\right),$$

$$\frac{\partial u_4}{\partial x_4} + u_k \frac{\partial u_4}{\partial x_k} + \lambda u_4 \frac{\partial u_k}{\partial x_k} = 0, \qquad j, k = 1, 2, 3, \tag{1}$$

where u_1, u_2, u_3 are the projections of the velocity on the coordinate axes x_1, x_2, x_3; a is the speed of sound; γ is the ratio of specific heats; x_4 is time; summation is performed over indices that occur twice.

A wave of order q is a solution of system (1), $u_i(x_1, \dots, x_4)$, where $i = 1, \dots, 4$, satisfying $4 - q$ functional dependencies

$$u_m = \psi_m(u_1, \dots, u_q), \qquad m = q + 1, \dots, 4. \tag{2}$$

In works [(1–6)], waves of first order and of order one less than the number of independent variables were investigated. In work [(7)], double waves for potential gas flow were studied. In the present note, by means of a modified method of work [(4)], double waves are considered without the assumption of potentiality of the flow. The main attention is devoted to obtaining an exact solution of the equations of spatial unsteady gas flow containing three arbitrary functions.

For the case under consideration, it follows from (2) that the functions $u_1, \dots, u_4$ possess common two-dimensional level surfaces. In the note the case is considered in which the level surfaces are two-dimensional planes, i.e.,

$$x_n + a_{mn} x_m + a_{0n} = 0, \qquad m = 3, 4, \tag{3}$$

where a_{mn} and a_{0n} are functions of u_1, u_2. Here and below, indices not specially specified take the values 1, 2.

From the condition that $\operatorname{grad} f$ is perpendicular to the level plane, where f is an arbitrary function of $u_1, \dots, u_4$, we obtain

$$\frac{\partial f}{\partial x_m} = a_{mn}\frac{\partial f}{\partial x_n}, \qquad m = 3, 4. \tag{4}$$

Using (2) and (4), we eliminate from (1) u_m and the derivatives of $u_1, \dots, u_4$ with respect to x_n. We obtain the system

$$A_i = 0, \qquad i = 1, \dots, 4. \tag{5}$$

Instead of system (5), in what follows we consider the equivalent system

$$B_n \equiv b_{n\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta} \equiv A_n = 0,$$

$$B_3 \equiv b_{3\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta} \equiv A_n(w\theta_n + \lambda w_n \theta) - A_4 w = 0, \tag{6}$$

$$B_4 \equiv b_{4\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta} \equiv A_n(w w_n + \theta_n) - A_3 w = 0,$$

where $w = \psi_3$; $\theta = \psi_4$; $w_n = \partial w/\partial u_n$; $\theta_n = \partial\theta/\partial u_n$.

Since equations (6) must be satisfied for any x_m, it is necessary to adjoin to system (6) the equations

$$\frac{\partial B_i}{\partial x_m} \equiv -I(-1)^{\alpha+\beta} b_{i\alpha\beta}\frac{\partial a_{m,3-\beta}}{\partial u_{3-\alpha}} \equiv -I L_{im} = 0, \tag{7}$$

where

$$I = \frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2} - \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1},$$

since $I = 0$ leads to the flows (3).

It is proved that the conditions $\partial^s B_i/\partial x_m^s = 0$, where $s = 2, 3, \dots$, are satisfied by virtue of (6) and (7).

Let us consider different gas flows depending on the value of the rank r of the matrix consisting of the coefficients of equations (6).

$$\begin{array}{cccc} \xi_1 & \xi_2-\theta_2 & \theta_2 & 0 \\ 0 & \theta_1 & \xi_1-\theta_1 & \xi_2 \\ \lambda w_1\theta\eta_1+c_{11} & \lambda w_1\theta\eta_2+c_{12} & \lambda w_2\theta\eta_1+c_{21} & \lambda w_2\theta\eta_2+c_{22} \\ \eta_1\theta_1 & \eta_2\theta_1 & \eta_1\theta_2 & \eta_2\theta_2 \end{array} \qquad (8)$$

$$\eta_n = a_{2n} + u_n + ww_n + \theta_n; \qquad \xi_n = a_{1n}w + a_{2n} + u_n + \theta_n;$$

$$c_{nn} = w\theta_n^2 - \lambda w\theta(1+w_n^2); \qquad c_{np} = w\theta_n\theta_p + \lambda\theta(w_n\theta_p - w_p\theta_n) - \lambda ww_{nw}p\theta$$

$$(n \neq p).$$

If $r = 4$, then only the trivial flow $u_i = \text{const}$ is possible. If $r = 3$, it is proved that a two-parameter family of two-dimensional level planes intersects along a common straight line. For $r = 1$, only flows with constant sound speed are possible. The requirement $r = 2$ leads to a system of equations that can be regarded as an algebraic system with respect to η_n and ξ_n. The simplest solution $\eta_n = 0,\ \xi_n = 0$ leads to potential flows (7).

We shall consider in detail only the case when

$$\eta_n = ww_n + \frac{w}{v}\left[(-1)^{3-n}\theta_{3-n} + \theta_n w\right], \qquad \xi_n = 0, \qquad (9)$$

where

$$v = (-1)^{3-\alpha}w_\alpha\theta_{3-\alpha}; \qquad \omega^2 = -(1+w_\alpha w_\alpha) + \frac{1}{\lambda\theta}(\theta_\alpha\theta_\alpha + v^2).$$

For $r = 2$, system (6) will contain only two independent equations, for example with indices $i = 1$ and $i = 4$, while system (7) contains four.

From (8) and (9) we find a_{mn} and substitute into (7). After a number of simplifications we obtain an overdetermined system for determining $w(u_1, u_2)$ and $\theta(u_1, u_2)$

$$w_\alpha\theta_\alpha = 0, \qquad (10)$$

$$\lambda\theta - \theta_\alpha\theta_\alpha = 0, \qquad (11)$$

$$(-1)^{3-\beta}w_\alpha w_{3-\beta}w_{\alpha\beta} = 0 \qquad \left(w_{\alpha\beta} = \frac{\partial^2 w}{\partial u_\alpha \partial u_\beta}\right). \qquad (12)$$

From (10) and (11), after first differentiating them with respect to u_1, u_2, we find

$$(-1)^{\alpha+\beta} w_{3-\alpha} w_{3-\beta} w_{\alpha\beta} = 0. \tag{13}$$

From (12) and (13) we obtain

$$w_2 w_{1n} - w_1 w_{n2} = 0, \tag{14}$$

which is equivalent to

$$\frac{w_1}{w_2} = k = \text{const}. \tag{15}$$

From (15) it follows that

$$w = \varphi(ku_1 + u_2), \tag{16}$$

where φ is an arbitrary function.

From (10) and (11) we find

$$\theta = \frac{\lambda}{4(1+k^2)}(u_1 - ku_2 + c)^2. \tag{17}$$

Substituting (9), (16), (17) into (6), we obtain

$$\frac{\partial^2 \Phi}{\partial x_1^2} + \left(\frac{1}{k} - k\right)\frac{\partial^2 \Phi}{\partial x_1 \partial x_2} - \frac{\partial^2 \Phi}{\partial x_2^2} = 0 \qquad \left(\frac{\partial \Phi}{\partial x_n} = u_n\right). \tag{18}$$

From (18) we find

$$\Phi = F(x_1 - kx_2) + G(kx_1 + x_2), \tag{19}$$

where F, G are arbitrary functions.

Using (3), it is easy to find the flow in four-dimensional phase space.

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Received
15 VI 1960

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Note: Figure translations are in progress. See original paper for figures.