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L. A. DIKII

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Abstract

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L. A. DIKII

STABILITY OF PLANE-PARALLEL FLOWS OF AN IDEAL FLUID*

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A flow is considered stable if an arbitrary small perturbation that has arisen at some initial moment remains bounded as time changes. In the opposite case the flow is unstable. Thus, in general, the investigation of stability requires solving the Cauchy problem for the equations of motion. The usual procedure consists in representing the solutions of an arbitrary Cauchy problem as a superposition of special solutions having the form of plane propagating waves. In this case the flow is regarded as unstable if there are wave solutions with unboundedly increasing amplitude; otherwise the flow is stable. This new definition of stability is not entirely equivalent to the original one. It may happen that not every solution of the equation of motion can be decomposed into plane waves. Moreover, sometimes the equation admits no wave solutions at all, whereas the Cauchy problem with arbitrary initial data is solvable. In such a case it becomes unclear whether the absence of growing wave solutions ensures stability, i.e., boundedness of any solution. It is precisely such an indeterminate situation that we encounter in investigating the stability of an inviscid fluid.

For the stream function of the perturbation field, if it is sought in the form of a plane harmonic wave $\varphi(z)e^{ia(x-ct)}$ (where z is the coordinate transverse to the basic flow, x along it), the equation

$$(U-c)(\varphi''-a^2\varphi)-U''\varphi=0$$

holds, where $U(z)$ is the profile of the basic flow (this is the Orr–Sommerfeld equation for zero viscosity, in degenerate form). For fixed a the problem has, generally speaking, only a finite number of eigenvalues c and, correspondingly, a finite number of eigenfunctions $\varphi(z)$. Therefore the solution of not every Cauchy problem for the equations of motion can be decomposed into waves. Thus, the question of whether the absence of waves with growing amplitude (complex c) implies true stability of the flow requires special investigation. In particular, is stability necessary when the velocity profile has no inflection point? (By Rayleigh' s theorem, in this case there can be no solutions with complex c.) Let us also note that the very formulation of the problem of finding wave solutions for an inviscid fluid proves to be indeterminate, since the above degenerate Orr–

Sommerfeld equation has a singular point where $U - c = 0$; consequently, its solutions are multivalued, and the question of a reasonable choice of branch remains open.

The circumstances noted have compelled one to abandon the study of fluid stability if it is assumed inviscid from the very beginning. Some rigorous results are usually obtained only by introducing viscosity into the equations, however small. This entails a considerable increase in analytic difficulties (it suffices to note that the complete

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the Orr–Sommerfeld equation has fourth, not second, order). The limiting transition in wave solutions as the viscosity tends to zero turns out to be very complicated. Almost all solutions disappear completely, while the remaining ones, as a rule, tend to discontinuous ones (the phenomenon of an internal boundary layer).

Nevertheless, it is possible to construct a closed theory of stability for an inviscid fluid without at all bringing viscosity into consideration. One need only abandon the study of wave solutions, bearing in mind that this is only one of the methods, and in the present case a poorly suited one, for solving the Cauchy problem on the development of arbitrary small disturbances that have arisen at some initial instant in the fluid. Thus, the basis is taken to be the equation of motion for the stream function, including time, from which the degenerate Orr–Sommerfeld equation is obtained by separation of variables.

First of all, it turns out that the Cauchy problem for this equation is always solvable. Further, the solution admits investigation for stability, i.e., for boundedness as $t \to \infty$. The main result is that, under quite general conditions, stability can in fact be violated only in those cases when there are unstable wave solutions (complex eigenvalues c of the Orr–Sommerfeld equation or multiple real ones). For smooth initial conditions the solution also turns out to be smooth. If we were first to introduce a small viscosity and then follow the limiting behavior of the solution of the Cauchy problem as this viscosity tends to zero, then this solution would tend continuously to the solution of the "inviscid" equation. In this process, near solid walls, boundary layers would form as usual, but inside the domain no "internal boundary layers" could form*.

Let us outline the proof of the assertions just stated. Thus suppose there is a plane-parallel flow between two solid walls. The velocity is parallel to the walls and depends only on the transverse coordinate: $U = U(z)$. For the stream function of the velocity field we have the vorticity equation

$$\frac{\partial \Delta\psi}{\partial t} + \frac{\partial(\Delta\psi, \psi)}{\partial(x, z)} = 0.$$

If this equation is linearized with respect to the disturbance field superposed on the mean flow, and the stream function of the disturbance field is sought in the form $e^{i\alpha x}\zeta(z,t)$, then for ζ we shall have

$$\left(-\frac{i}{\alpha}\frac{\partial}{\partial t} + U(z)\right)\left(\frac{\partial^2}{\partial z^2} - \alpha^2\right)\zeta(z,t) - U''(z)\zeta(z,t) = 0. \tag{1}$$

The boundary conditions are then

$$\zeta(a,t) = \zeta(b,t) = 0.$$

We take as given the initial condition

$$\zeta(z,0) = \zeta_0(z).$$

We assume the functions $U(z)$, $\zeta_0(z)$ to be analytic in a neighborhood of the interval $[a,b]$. We subject the equation to the Laplace transform with respect to t:

$$(U(z) - c)\left(\frac{\partial^2}{\partial z^2} - \alpha^2\right)\zeta^*(z,c) - U''(z)\zeta^*(z,c) = f(z), \tag{3}$$

where

$$\zeta^*(z,c) = \int_0^\infty e^{i\alpha ct}\zeta(z,t)\,dt, \qquad f(z) = -\frac{i}{\alpha}(\zeta_0'' - \alpha^2\zeta_0).$$

* *Note added in proof.* When the present paper had already been submitted for publication, we became acquainted with work (3), where similar assertions are made.

This equation differs from the degenerate Orr–Sommerfeld equation in that it is nonhomogeneous. It must be solved for values of c in the half-plane $\operatorname{Im} c > 0$, after which the solutions of (1) can be found by means of the inversion formula for the Laplace transform.

We construct the Green' s function for equation (2) by means of two solutions $\xi_1^*(z,c)$ and $\xi_2^*(z,c)$ of the corresponding homogeneous equation with the initial conditions

$$\xi_1^*(a,c) = \xi_2^*(b,c) = 0, \qquad \xi_1^{*\prime}(a,c) = \xi_2^{*\prime}(b,c) = 1.$$

Let $W(c) = \xi_1^*(b,c)$. Then

$$\xi^*(z,c) = \int_0^b G^*(z,z';c) f(z')\, dz',$$

where

$$G^*(z,z';c) = \begin{cases} \xi_1^*(z,c)\xi_2^*(z',c)/W(c)[U(z')-c], & z < z', \\ \xi_1^*(z',c)\xi_2^*(z,c)/W(c)[U(z')-c], & z > z'. \end{cases}$$

It is easily shown that $G^*(z,z';c)$ is analytic in c in the upper half-plane, except possibly for a finite number of poles (where $W(c) = 0$). As $|c| \to \infty$ in this half-plane it decreases as $O(|c|^{-1})$. Hence, according to the known theorems on the Laplace transform, it follows that $G^*(z,z';c)$ is the Laplace transform of some function $G(z,z';t)$:

$$G(z,z';t) = \frac{\alpha}{2\pi} \int_{\gamma i - \infty}^{\gamma i + \infty} e^{-i\alpha c t} G^*(z,z';c)\, dc, \qquad \gamma > 0, \tag{3}$$

and the solution of equation (1) can be written in the form

$$\xi(z,t) = \int_a^b G(z,z';t) f(z')\, dz'.$$

Thus the existence of a solution of the Cauchy problem has been proved. From the very construction of the solution there follows also its uniqueness in the class of functions growing no faster than exponentially. The smoothness of the solution under smooth initial conditions is also obvious.

Let us turn to the question of stability, i.e., of the boundedness of the solution $\xi(z,t)$ as $t \to \infty$. Formula (3) is still of little use for studying the behavior of $G(z,z';t)$ as $t \to \infty$, since as $t \to \infty$ the factor $e^{-i\alpha c t}$ grows rapidly, because $\operatorname{Im} c > 0$, and boundedness of the integral can occur only through interference of the exponentials for different c, the latter being difficult to investigate. Therefore we shall try to lower the contour of integration downward as far as proves possible. For this the function $G^*(z,z';c)$ must be analytically continued from the upper half-plane, in which it was initially defined, into a neighborhood of the real axis. It is not hard to show that the only singular points of the functions $\xi_1^*(z,c)$, $\xi_2^*(z,c)$, and $W(c)$ entering the expression for G^* are the points $c = U(a)$, $U(b)$, and $U(z)$, as well as $U(\tilde{z})$ for those $\tilde{z}$ for which $U'(\tilde{z}) = 0$.

If c_0 is a real number distinct from those just named, then by $\xi_1^*(z,c_0)$ and $\xi_2^*(z,c_0)$ we shall understand solutions of the equation, analytic along the segment $[a,b]$ deformed in neighborhoods of those points z_0 where $U(z_0) = c_0$, in the following way. If $U'(z_0) > 0$, then the point z_0 is bypassed from below; if $U'(z_0) < 0$, then from above. It is precisely under this definition

$\xi^*_{1,2}(z, c_0)$ becomes the limit of $\xi^*_{1,2}(z, c)$ as $c \to c_0$ from the upper half-plane.

The function $G^*(z, z'; c)$ may have, in addition to the singular points $c = U(a)$, $U(b)$, $U(z)$, $U(z')$, $U(\bar{z})$ (we shall call them points "of the first kind"), also poles where $W(c) = 0$. These values of c are eigenvalues for the Orr–Sommerfeld equation. The corresponding eigenfunction is $\xi^*_1(z, c)$. If c is real, then on the axis there may also be singular points. $\xi^*_1(z, c)$ must be analytic in z along a contour bypassing the singular points, as was indicated above. (It is curious to note that our definition of an eigenfunction with bypassing of singular points coincides with that given by Tollmien–Lin from entirely different considerations, proceeding from the requirement that the "inviscid" eigenfunctions be limits of "viscous" ones.)

Next we impose the only restriction: we require that the values c equal to $U(a)$, $U(b)$, $U(\bar{z})$ not be eigenvalues. In this case it can be proved that there can be only a finite number of eigenvalues. Furthermore, one can estimate the modulus $|G^*(z, z'; c)|$ near singular points. It is proved that if c_0 is a singular point of the first kind, then $|G^*(z, z'; c)| < K|c - c_0|^{-1}$.

Now we can replace the contour of integration in formula (3) by a contour lying entirely in the lower half-plane, except for loops enclosing the singular points of the first kind. The radius of these loops is chosen smaller than K/t. In addition to the integral, there will appear a finite number of further terms—the residues at those points c where $W(c) = 0$. It is easy to show that the integral is bounded as $t \to \infty$. As for the residues, unboundedness appears here only if the pole c is complex, or if it is real and multiple, i.e., if $d^{lW}(c)/dc^l = 0$, $l = 0, \dots, m$. In the latter case, terms are singled out which tend to infinity as a power as $t \to \infty$. Thus, instability appears in fact only on account of unstable wave solutions.

Let us note that the present paper is close in its main idea to the work of K. V. Brushlinskii ([1]), which deals with equations of another type. There, too, the solution of an equation containing time cannot be expanded in wave solutions because the system of such solutions is incomplete; the apparatus of the Laplace transform is also applied, and the same conclusion is reached: stability can be violated only because of unbounded wave solutions, despite the fact that, generally speaking, there are few wave solutions. The formal difference between our equations and the equations considered in ([1]) consists in the fact that in ([1]) the integral in the inversion formula for the Laplace transform could not be replaced by a sum of residues (i.e., completeness of the system of wave solutions was not obtained) because of an insufficient degree of decrease of the Green' s function in the lower half-plane, whereas in our case this is caused by the presence in the Green' s function of branch points (caused by singularities of the coefficients of the equation). The greatest analytical difficulty for us is precisely the estimate of the growth of the Green' s function near these branch points.

Let us add further that one example of a study of stability for an inhomogeneous fluid in the presence of density stratification is considered in the work ([2]).

Institute of Physics of the Atmosphere
Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.