MATHEMATICS

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1960

SovietRxiv

Abstract

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ON THE COINCIDENCE OF THE DIMENSIONS $\operatorname{ind} G$ AND $\dim G$ FOR LOCALLY BICOMPACT GROUPS

(Presented by Academician P. S. Aleksandrov on 26 II 1960)

B. Pasynkov proved ([1]) the coincidence of the dimensions $\dim G$, $\operatorname{ind} G$, and $\operatorname{Ind} G$ for an arbitrary bicompact group G. In the present note the relation $\operatorname{ind} G = \dim G$ is proved for an arbitrary locally bicompact group. An important role in finding this fact is played by Theorem 1, which is also of independent interest.

Theorem 1. *A locally bicompact group is strongly paracompact.*

Proof. Let G be a locally bicompact group and let V be a neighborhood of the identity in it with bicompact closure. Put

$$E = \bigcup_{n=1}^{\infty} V^n,$$

where V^n is the n-th power of the set V in the group G.

E is an open and, consequently, closed ([2]) subgroup of the group G. We shall need the following two facts:

1) The set $[V]^n$ is bicompact for every positive integer n.
2) The relation $[V]^n \subset [V^n]$ holds.

Both these assertions follow from the continuity of the mapping f of the direct product space of n copies of the group G into the group G, defined by the formula

$$f(a_1 \times \cdots \times a_n) = a_1 \cdot \ldots \cdot a_n.$$

From the relations

$$E = [E] \supset \bigcup_{n=1}^{\infty} [V^n] \supset \bigcup_{n=1}^{\infty} [V]^n \supset E$$

it follows that

$$\bigcup_{n=1}^{\infty} [V]^n = E,$$

i.e. E is the union of a countable set of bicompacta and therefore is finally compact.

By a theorem of Yu. M. Smirnov ([3]), every finally compact space is strongly paracompact.

Thus, E is strongly paracompact. Since E is an open-and-closed subgroup of the group G, it follows easily that the group G itself is strongly paracompact. (We note that the group G itself need not be finally compact.)

Corollary 1. A locally bicompact topological group is normal.

Corollary 2. For a finite-dimensional locally bicompact topological group,

$$\dim G \leq \operatorname{ind} G.$$

This follows from the fact that the relation $\dim X \leq \operatorname{ind} X$ is valid for an arbitrary strongly paracompact space X ([6]).

Theorem 2. *For an arbitrary locally bicompact topological group G, the equality*

$$\dim G = \operatorname{ind} G$$

holds.

Proof. The inequality $\dim G \leq \operatorname{ind} G$ has been proved by us. B. Pasynkov drew my attention to the fact that the relation $\operatorname{ind} G \leq \dim G$ for locally bicompact topological groups is a trivial consequence of the theorem proved in ([4]): a locally bicompact

a finite-dimensional group is locally isomorphic to the direct product of a cube E^n in Euclidean space of the corresponding dimension and a zero-dimensional bicompactum.

Since there exist locally bicompact spaces that are not paracompact, Theorem 2 implies:

Corollary 3. Not every locally bicompact space can be embedded as a closed subset in the space of a locally bicompact topological group.

The assertion of Corollary 3 is interesting because, by a well-known theorem of Markov, a completely regular space can always be embedded as a closed subset in the space of some topological group, namely in the space of the free topological group of this space. On the other hand, every bicompactum can obviously be embedded in the space of some bicompact group (for example, in a product of a sufficiently large number of circles).

Theorem 3. *The free topological group of a bicompactum is finally compact and hence strongly paracompact.*

Proof. It is known ([5]) that the space of the free topological group is represented as a sum of a countable number of bicompacta; from this its final compactness follows directly and, by the aforementioned theorem of Yu. M. Smirnov, its strong paracompactness.

We note that Theorem 3 is a strengthening of a result of M. I. Graev, who proved that the free topological group of an arbitrary bicompactum is normal ([5]).

In conclusion I express my sincere gratitude to Academician P. S. Aleksandrov for posing the problems and for his guidance.

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Received
20 II 1960

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Note: Figure translations are in progress. See original paper for figures.