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Abstract

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ON INVARIANT SURFACES OF A SYSTEM OF TWO DIFFERENTIAL EQUATIONS

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Consider a system of two differential equations

$$\begin{aligned} \frac{dx}{dt} &= f_1(x,y,t) + \mu R_1(x,y,t,\mu), \\ \frac{dy}{dt} &= f_2(x,y,t) + \mu R_2(x,y,t,\mu), \end{aligned} \tag{1}$$

where the functions f_1, f_2, R_1, and R_2 are continuous, periodic in t with period ω, and uniformly analytic in x, y in a neighborhood of each point x, y for $t \in [0, \omega]$ and sufficiently small $|\mu|$.

We shall assume that for $\mu = 0$ system (1) has an invariant surface M_0 homeomorphic to a torus, and we pose the question of when one can specify a $\mu_0 > 0$ such that, for $|\mu| < \mu_0$, system (1) has an invariant surface M_μ. An analogous question was solved by Levinson (^{1}), Diliberto and Hufford (^{2}), Marcus (^{3}), and Coddington (^{4}). All these authors assumed that either the rotation number on the toroidal surface M_0 is irrational, or all solutions situated on this surface are periodic. We shall consider the case in which the rotation number on M_0 is rational and not all solutions situated on M_0 are periodic.

It is natural to consider system (1) in the toroidal space obtained by identifying the planes $t = n\omega$ $(n = 0, \pm 1, \pm 2, ...)$.

Let

$$x = \varphi(x_0, y_0, t, \mu), \qquad y = \psi(x_0, y_0, t, \mu) \tag{2}$$

be the solution of system (1) with initial data $x = x_0,\ y = y_0$ at $t = 0$. Suppose that the solution of system (1) with initial data $x = x_0,\ y = y_0$ at $t = 0$ can be continued to the interval $0 \le t \le \omega$. Associate to the point x_0, y_0 the point $x = \varphi(x_0, y_0, \omega, \mu),\ y = \psi(x_0, y_0, \omega, \mu)$. The transformation obtained in this way will be denoted by I_μ.

In what follows we shall assume that there exists a closed Jordan curve Γ_0, without self-intersections, invariant with respect to the transformation I_0. Through the curve Γ_0 draw all possible integral curves of system (1) for $\mu = 0$; then we obtain an invariant surface M_0, homeomorphic to a torus. Suppose that on M_0 there are closed integral curves (periodic solutions) of system (1) for $\mu = 0$. It is known (^{5,6}) that all these solutions have a common period $k\omega$ (k natural). We shall assume that these periodic solutions have no characteristic exponents equal to zero. In addition, we shall suppose that the invariant surface M_0 is asymptotically stable (for stability of invariant sets, see (^{7})).

1. Let us clarify somewhat more fully the structure of the surface M_0. Note that on M_0 there can be only a finite number of closed integral curves. From the stability of the set M_0 it follows that one of the characteristic exponents of any periodic solution lying on M_0 is negative.

To each periodic solution there correspond on Γ_0 k points fixed with respect to I_0^k. We shall say that a point p, fixed with respect to I_0^k, is unstable if, for every point $q \in \Gamma_0$ lying sufficiently close to p, the relation

$$I_0^{-kn}(q) \longrightarrow p \quad n \to +\infty \tag{3}$$

holds.

In the opposite case we shall call the fixed point p stable. For a stable point the relation

$$I_0^{kn}(q) \longrightarrow p, \quad n \to +\infty, \tag{4}$$

holds if $q \in \Gamma_0$ and q is situated sufficiently close to p. It is not difficult to see that on Γ_0 the stable and unstable fixed points alternate.

Let $p_j = (x_j, y_j)$ be a point of the curve Γ_0 unstable and fixed with respect to I_0^k. Then one of the characteristic exponents of the periodic solution $x = \varphi(x_j, y_j, t, 0)$, $y = \psi(x_j, y_j, t, 0)$ is positive. It can be shown that the roots of the characteristic equation corresponding to this periodic solution are positive. Then$^{(8)}$, in a neighborhood of the solution $x = \varphi(x_j, y_j, t, 0)$, $y = \psi(x_j, y_j, t, 0)$, system (1) can be represented in the form

$$\frac{d\xi}{dt} = \lambda_1\xi + F_1(\xi, \eta, t), \quad \frac{d\eta}{dt} = \lambda_2\eta + F_2(\xi, \eta, t), \tag{5}$$

where $\lambda_1 > 0$, $\lambda_2 < 0$; F_1 and F_2 are series in powers of ξ and η with continuous and $k\omega$-periodic coefficients.

A. M. Lyapunov$^{(8)}$ proved that, in a neighborhood of the solution $\xi = \eta = 0$, system (5) has a one-parameter family of solutions

$$\xi = ce^{\lambda_1 t} + \sum_{m=2}^{\infty} L_m(t)c^m e^{m\lambda_1 t}, \quad \eta = \sum_{m=2}^{\infty} M_m(t)c^m e^{m\lambda_2 t}, \tag{6}$$

where the functions $L_m(t)$ and $M_m(t)$ are periodic, and the series on the right in (6) converge uniformly for sufficiently small $|c|$ and all $t \leq 0$.

Consider the curve Λ_j of the form

$$\xi = c + \sum_{m=2}^{\infty} L_m(0) c^m, \quad \eta = \sum_{m=2}^{\infty} M_m(0) c^m \quad \text{for } |c| \leq \alpha, \tag{7}$$

where $\alpha > 0$ is sufficiently small. It is not difficult to prove that the curve Λ_j lies on Γ_0. Introduce the set

$$\mathcal{H}_j = \sum_{s=0}^{\infty} I_0^s(\Lambda_j). \tag{8}$$

This set consists of k open arcs lying on Γ_0. By their ends these arcs adjoin stable points fixed with respect to I_0^k. If, besides the solution passing through p_j, system (1) for $\mu = 0$ has no periodic solutions on M_0 with positive characteristic exponents, then the closure of the set $\mathcal{H}_j$ coincides with Γ_0. If, however, such solutions exist, then for each of them one should construct a set $\mathcal{H}_j$, and the closure of the sum of all these sets will give Γ_0.

2. **Theorem 1.** *Suppose that:*

1) for $\mu = 0$ system (1) has an invariant surface M_0, homeomorphic to a torus;

2) the surface M_0 is asymptotically stable;

3) on the surface M_0 there are periodic solutions;

4) both characteristic exponents of each of the periodic solutions lying on M_0 are nonzero.

Then there exists a $\mu_0 > 0$ such that, for $|\mu| \leqslant \mu_0$, system (1) has an invariant surface M_μ, homeomorphic to a torus.

The assumptions we have made make it possible to prove not only the existence of the surface M_μ, but also its stability.

Theorem 2. If the assumptions of Theorem 1 are satisfied, then there exists a $\mu_0 > 0$ such that, for $|\mu| \leqslant \mu_0$, the surface M_μ is asymptotically stable.

3. Let us investigate the question of the structural stability of system (1) for $\mu = 0$. As follows from the results of E. A. Barbashin (^{7}), under the assumptions made there exist two smooth surfaces S_1 and S_2, homeomorphic to a torus and such that: a) S_2 lies inside the region bounded by S_1; b) all solutions of system (1) for $\mu = 0$, as t increases, enter into the region G bounded by the surfaces S_1 and S_2; c) the surface M_0 lies inside G; d) any solution of system (1) with $\mu = 0$, beginning in G, tends to M_0 as $t \to +\infty$.

Definition. We shall call system (1) with $\mu = 0$ **structurally stable** in G if, for every $\varepsilon > 0$, one can specify a $\delta > 0$ such that, for all $|\mu_0| < \delta$, there exists a topological transformation T of the region G onto itself with the following properties: 1) the distance between the points $p \in G$ and $T(p)$ is less than ε; 2) T transforms the integral curves of system (1) with $\mu = 0$ into the integral curves of system (1) with $\mu = \mu_0$.

We note that this definition is a modification of the definition given by A. A. Andronov and L. S. Pontryagin ([9]).

Theorem 3. If the conditions of Theorem 1 are satisfied, then system (1) with $\mu = 0$ is structurally stable in $\overline{G}$.

4. Under certain additional assumptions concerning the behavior of solutions of system (1) for $\mu = 0$, it is possible to prove the smoothness of the curve Γ_μ and, consequently, of the surface M_μ.

Let $q \in \Gamma_0$ be a stable fixed point with respect to the transformation I_0^k. Through the point q passes the periodic solution $x = \varphi(t)$, $y = \psi(t)$. Suppose that the roots of the characteristic equation corresponding to this solution are positive and distinct. Then, in a neighborhood of the solution $x = \varphi(t)$, $y = \psi(t)$, system (1) can be represented in the form (5), where $\lambda_2 < \lambda_1 < 0$. We shall call the direction corresponding, in the plane x, y $(t = 0)$, to the direction of the $O\xi$ axis the **principal direction**.

Theorem 4. Suppose that the conditions of Theorem 1 are satisfied. Suppose, in addition, that:

1) the characteristic equation corresponding to each stable periodic solution lying on M_0 has positive and distinct roots;

2) the curve Γ_0 is smooth;

3) at each stable invariant point of the curve Γ_0 with respect to I_0^k, the curve Γ_0 is tangent to the principal direction.

Then the surface M_μ is smooth.

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CITED LITERATURE

1. N. Levinson, *Ann. of Math.*, **52**, 727 (1950).

2. S. P. Diliberto, G. Hufford, *Contributions to the Theory of Nonlinear Oscillations*, **3**, 1956.

3. M. Marcus, ibid.

4. P. Koosis, ibid.

5. A. Lyapunov, *On Curves Defined by Differential Equations*, Moscow-Leningrad, 1947.

6. V. V. Nemytskii, V. V. Stepanov, *Qualitative Theory of Differential Equations*, Moscow-Leningrad, 1949.

7. E. A. Barbashin, *Matem. sbornik*, **29**, no. 2 (1951).

8. A. M. Lyapunov, *The General Problem of the Stability of Motion*, Moscow-Leningrad, 1950.

9. A. A. Andronov, L. S. Pontryagin, *DAN*, **14**, no. 5 (1937).

Note: Figure translations are in progress. See original paper for figures.

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