V. A. IL'IN and I. A. SHISHMAREV

In the present paper we consider the Dirichlet and Neumann problems and the eigenfunction problem for the operator

1960

SovietRxiv

Abstract

Full Text

MATHEMATICS

V. A. IL' IN and I. A. SHISHMAREV

SOME PROBLEMS FOR THE OPERATOR $Lu = \operatorname{div}[p(x) \operatorname{grad} u] - q(x)u$ WITH DISCONTINUOUS COEFFICIENTS

(Presented by Academician I. G. Petrovskii, 20 VI 1960)

In the present paper we consider the Dirichlet and Neumann problems and the eigenfunction problem for the operator $Lu = \operatorname{div}[p(x) \operatorname{grad} u] - q(x)u$ with discontinuous coefficients.

1°. Let there be given an open N-dimensional domain g with boundary Γ, and inside it an $(N-1)$-dimensional surface C, homeomorphic to a sphere, which divides g into subdomains g_1 and g_2. Let T be an open domain containing within itself the closed domain $(g+\Gamma)$. Consider in the domain $(g+\Gamma)$ the following Dirichlet problem:

$$L_1 u = \operatorname{div}[p_1(x) \operatorname{grad} u] - q_1(x)u = f_1(x) \quad \text{in } g_1,$$

$$L_2 u = \operatorname{div}[p_2(x) \operatorname{grad} u] - q_2(x)u = f_2(x) \quad \text{in } g_2, \tag{1}$$

$$u\big|_\Gamma = \varphi(x), \qquad [u]\big|_C = \psi(x), \qquad \left[p\frac{\partial u}{\partial n}\right]\bigg|_C = \chi(x),$$

where

$$[u]\big|_C \equiv u\big|_{C-0} - u\big|_{C+0}; \qquad \left[p\frac{\partial u}{\partial n}\right]\bigg|_C = p_1\frac{\partial u}{\partial n}\bigg|_{C-0} - p_2\frac{\partial u}{\partial n}\bigg|_{C+0};$$

n is the exterior normal for the domain g_1; the symbols $C-0$ and $C+0$ mean that the limiting values are taken, respectively, from the inner and the outer side of the surface C relative to g_1.

Definition 1. By a classical solution of the Dirichlet problem (1) we shall mean a function $u(x)$ satisfying the following conditions:
1) $u(x)$ belongs to the class $^*C^{(0)}$ in each of the closed domains (g_1+C) and $(g_2+C+\Gamma)$; to the class $C^{(1)}$ in each of the domains (g_1+C) and (g_2+C); to

the class $C^{(2)}$ in each of the open domains g_1 and g_2;
2) $u(x)$ satisfies, in the usual classical sense, all the conditions of problem (1).

Suppose the following 5 conditions are fulfilled:
1) the surface C belongs to the Lyapunov class, and the surface Γ is regular**;
2) the functions $p_i(x)$, $q_i(x)$, $f_i(x)$ $(i = 1, 2)$ are defined and belong to the classes: $p_1(x) \in C^{(1,\mu)}$ in the domain $(g_1 + C)$; $p_2(x) \in C^{(1,\mu)}$ in $(T - g_1)$; $q_1(x) \in C^{(0,\mu)}$ in $(g_1 + C)$; $q_2(x) \in C^{(0,\mu)}$ in $(T - g_1)$; $f_1(x) \in C^{(0,\mu)}$ in g_1; $f_2(x) \in C^{(0,\mu)}$ in g_2; moreover, $f_1(x) \in C^{(0)}$ in $(g_1 + C)$; $f_2(x) \in C^{(0)}$ in $(g_2 + C + \Gamma)$;
3) $p_i(x) > 0,\ q_i(x) \geqslant 0$ $(i = 1, 2)$ everywhere in the domain of their definition;
4) the function $\varphi(x)$ is defined and continuous on the surface Γ;
5) the functions ψ and χ are defined on the surface C and belong on it to the classes: $\psi \in C^{(1,\mu)}$, $\chi \in C^{(0,\mu)}$.

* The classes $C^{(n)}$, $C^{(n,\mu)}$ used in this article are defined in the book [1].
** The surface Γ is called regular if, in the domain g bounded by this surface, the Dirichlet problem for the Laplace equation is solvable for every continuous boundary function.

The five conditions stated will be called **conditions A**. We have proved the following two assertions:

Theorem 1 (uniqueness). *Suppose that the first and third of conditions A are satisfied. Then there can exist only one classical solution of the Dirichlet problem (1).*

Theorem 2 (existence). *If conditions A are satisfied, then there exists a (moreover unique) solution of the Dirichlet problem (1), and this solution belongs to the class $C^{(1,\mu)}$ in each of the domains $(g_1 + C)$ and $(g_2 + C)$.*

In recent papers by O. A. Oleinik ([2],[3]), existence theorems were proved for the solution of the Dirichlet problem for a general linear elliptic equation with discontinuous coefficients, but under the assumption that the boundaries of the domains and the coefficients of the equation satisfy very stringent smoothness requirements (in comparison with conditions A), which grow without bound as the number N of dimensions increases.

Remark 1. Theorems 1 and 2 extend to the case when the operators L_i $(i = 1, 2)$ have the form:

$$L_i u = \operatorname{div}[p_i(x) \operatorname{grad} u] - \sum_{k=1}^{N} b_{ik}(x) \frac{\partial u}{\partial x_k} - q_i(x) u,$$

where $b_{1k} \in C^{(0,\mu)}$ in $(g_1 + C)$, $b_{2k} \in C^{(0,\mu)}$ in $(T - g_1)$.

Remark 2. Theorems 1 and 2 carry over to the case when inside the surface Γ there lie n closed surfaces of discontinuity of the coefficients $C_1, C_2, \dots, C_n$, belonging to the Lyapunov class, on each of which its own boundary conditions are prescribed:

$$[u]\big|_{C_i} = \psi_i, \qquad \left[p\frac{\partial u}{\partial n}\right]\bigg|_{C_i} = \chi_i.$$

Here some of the surfaces C_i may lie inside others.

Remark 3. Results analogous to those formulated above have also been obtained by us for the Neumann problem, i.e. for a problem of the form (1) in which the condition $u|_\Gamma = \varphi$ is replaced by

$$\left(p_2\frac{\partial u}{\partial n_2} + hu\right)\bigg|_\Gamma = 0,$$

where $h(x) \geq 0$; n_2 is the outward normal for the domain g.

If to the definition of a classical solution one adds that $u(x) \in C^{(1)}$ in the closed domain $(g_2 + C + \Gamma)$, and to conditions A one adds: 1) Γ is a surface of Lyapunov type; 2) $h(x)$ is continuous on Γ, then for the Neumann problem theorems textually coinciding* with Theorems 1 and 2 will be valid.

2°. The connection of the classical solution of the Dirichlet problem (1), for $\varphi = \psi = \chi = 0$, with the so-called generalized solution of this problem** has been studied. By the method set forth in paper ([6]), it has been proved that the classical solution of the indicated problem is at the same time a generalized one, i.e. it has been proved that the classical solution has first derivatives square-integrable over the closed domain $(g + \Gamma)$.

3°. The Green' s function $K(x, y)$ of problem (1) with discontinuous coefficients has been constructed, its symmetry*** has been proved, and the following two properties of this function have been established:

1) $K(x, y)$ is continuous as a function of the aggregate (x, y) everywhere in the closed domain $(g + \Gamma)$ for $x \neq y$ (including also the surface C!);

* If $h(x) \equiv 0$, the uniqueness theorem for the solution of the Neumann problem is valid up to an arbitrary constant term.

** The generalized solution is defined in the same way as in ([4],[5]).

*** For the proof of symmetry, the results of item 2° and a lemma proved in paper ([7]) are used essentially.

2) for $K(x, y)$ the following estimate, uniform in the closed domain $(g + \Gamma)$, is valid:

$$|K(x, y)| \leq c_1 + c_2 \ln\frac{1}{r_{xy}}, \qquad \text{for } N = 2,$$

$$|K(x,y)| \leq c_3 r_{xy}^{2-N}, \qquad \text{for } N > 2, \tag{2}$$

where c_1, c_2, c_3 are certain constants.

Let us emphasize that these results were obtained by us under condition A, i.e., under the condition that the surface Γ satisfies only the regularity condition and, generally speaking, is not smooth.

4°. The eigenfunction problem for an operator with discontinuous coefficients has been considered:

$$\begin{gathered} L_1 u + \lambda u = 0 \quad \text{in } g_1, \\ L_2 u + \lambda u = 0 \quad \text{in } g_2, \\ u|_\Gamma = 0, \qquad [u]|_C = 0, \qquad \left[p\frac{\partial u}{\partial n}\right]_C = 0. \end{gathered} \tag{3}$$

(here L_1 and L_2 have the same meaning as in (1)).

Definition 2. A **classical eigenfunction** of problem (3) will mean a function $u(x)$, not identically equal to zero, which: 1) satisfies condition 1) of Definition 1; 2) for some λ, in the usual classical sense, satisfies all the conditions of problem (2).

Relying on the properties of the Green' s function stated above, we have proved:

Theorem 3. *If the first three conditions of A are fulfilled, then there exists a complete (in $L_2(g)$) system of orthonormal classical eigenfunctions of problem (3), and each of these eigenfunctions, moreover, belongs to the class $C^{(1,\mu)}$ in each of the domains $(g_1 + C)$, $(g_2 + C)$.*

The system of classical eigenfunctions of problem (3) is equivalent to the system of eigenfunctions of the homogeneous integral equation

$$u(x) = \lambda \int_g K(x,y) u(y)\, dy, \tag{4}$$

whose kernel is the Green' s function constructed above. (For the proof of such an equivalence, the symmetry and the two properties of the Green' s function established above play a fundamental role.)

Remark 4. Theorem 3 carries over to the case where inside the surface Γ there lie n closed coefficient-discontinuity surfaces $C_1, C_2, \dots, C_n$ belonging to the Lyapunov class; on each of them the boundary conditions $[u]|_{C_i} = 0$ and $\left[p\frac{\partial u}{\partial n}\right]_{C_i} = 0$ are prescribed. In this case some of the surfaces C_i may lie inside others.

Remark 5. Analogous results have also been obtained by us for the eigenfunctions of the Neumann problem, i.e., for the eigenfunctions of problem (3) in which the condition $u|_\Gamma = 0$ is replaced by the condition

$$\left[p_2 \frac{\partial u}{\partial n_2} + hu\right]_\Gamma = 0,$$

where $h(x) \geq 0$. In this case it is additionally required that the surface Γ belong to the Lyapunov class and that $h(x)$ be continuous on Γ. To the definition of an eigenfunction there is added the requirement $u(x) \in C^{(1)}$.

in the closed domain $(g_2 + C + \Gamma)$, and a theorem completely analogous to Theorem 3 is proved.

5°. By the method set forth in paper ([6]), the following has been proved.

Theorem 4. *The complete system of classical eigenfunctions of problem (3) coincides with the complete system of generalized* eigenfunctions of this problem.*

Thus, every classical eigenfunction of problem (3) has first derivatives that are square-integrable over the closed domain $(g + \Gamma)$.

This assertion, established for any surface Γ satisfying only the regularity condition, is used essentially below in this paper, as well as in subsequent papers.

6°. We have studied the question of a uniform estimate for the eigenfunctions of problem (3). Relying on the results of item 5°, on the estimate of the Green function (2), and on the method proposed in §§2–4 of paper ([7]), we arrive at the following assertion:

Theorem 5. *Under the conditions of Theorem 4 there exists a constant c_0 such that, uniformly in the closed domain $(g + \Gamma)$, the inequality holds:*

$$|u_n(x)| \leqslant c_0 \lambda_n^{N/4} \tag{5}$$

(here $u_n(x)$ is any eigenfunction of problem (3) corresponding to the eigenvalue λ_n).

An estimate of the form (5) was obtained by D. M. Eidus ([8]) for the case of the Laplace operator and under the condition that the boundary surface Γ belongs to the Lyapunov class. In paper ([7]) an estimate of the form (5) was established by us for the case of an arbitrary self-adjoint elliptic operator with smooth coefficients, under the condition that Γ satisfies only the regularity condition. It has now been possible to establish estimate (5) also for an operator with discontinuous coefficients when the first three of conditions A are fulfilled, i.e., under the assumption that Γ satisfies only the regularity condition. It goes without saying that estimate (5) is also valid for the case when inside the surface Γ there lies any number n of surfaces of discontinuity of the coefficients (see Remark 4).

The authors express their gratitude to A. N. Tikhonov for discussion of the results of this work and for a number of valuable suggestions.

Moscow State University
named after M. V. Lomonosov

Received
18 VI 1960

CITED LITERATURE

[1] K. Miranda, *Partial Differential Equations of Elliptic Type*, IL, 1957.
[2] O. A. Oleinik, DAN, **124**, No. 6, 1219 (1959).
[3] O. A. Oleinik, UMN, **14**, 5, 164 (1960).
[4] R. Courant, D. Hilbert, *Methods of Mathematical Physics*, 2, Ch. 7, 1951.
[5] S. G. Mikhlin, *The Problem of the Minimum of a Quadratic Functional*, 1952.
[6] V. A. Il' in, I. A. Shishmarev, DAN, **126**, No. 6 (1959).
[7] V. A. Il' in, I. A. Shishmarev, Izv. AN SSSR, Ser. Mat., **24**, 6 (1960).
[8] D. M. Eidus, DAN, **90**, No. 6 (1953).

---

* For the definition of generalized eigenfunctions see ([4,5]).

Note: Figure translations are in progress. See original paper for figures.