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Abstract

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MATHEMATICS

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L-NETS AND L-SEQUENCES OF AN N-DIMENSIONAL PROJECTIVE SPACE

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1. Let

$$\ldots,(A_{-n}),\ldots,(A_{-1}),\ (A_0),\ (A_1),\ldots,(A_n),\ldots \tag{A}$$

be a Laplace sequence of the N-dimensional projective space P_N, generated by the net (A_0). In the case $N = 3$, S. P. Finikov considered Laplace sequences formed by harmonic Wilczynski nets, i.e., such sequences for which the corresponding rays of the congruences $(A_{i+1}A_{i+2})$ and $(A_{i-1}A_{i-2})$ intersect (1). In this case it is not difficult to show that the indicated points of intersection themselves describe a Laplace sequence, likewise consisting of harmonic Wilczynski nets.

In our works, L-nets and L-sequences of the space P_4 have been constructed; these serve as a generalization of harmonic Wilczynski nets and of the Laplace sequences constructed on such nets (2). A direct connection has been established between L-sequences in P_4 and Laplace sequences of harmonic Wilczynski nets in P_3, which appears under the mapping of an L-sequence onto a fixed hyperplane of the space P_4 (3).

In the present article, the generalization of harmonic Wilczynski nets and of Laplace sequences from such nets is carried out for an N-dimensional projective space P_N.

2. The axis of the conjugate net (A_i) of an odd-dimensional projective space P_{2n+1} at the point A_i will be called the line along which the $(n+1)$-dimensional tangent planes of the lines of the net passing through the point A_i intersect. We shall call the conjugate net (A_i) a net $L^{2n+1}_{n,n}$ if, for each of its points, the planes $[A_{i+1}A_{i+2}\ldots A_{i+n+1}]$ and $[A_{i-1}A_{i-2}\ldots A_{i-n-1}]$ intersect (here the lower indices denote the dimensions of the intersecting planes, and the upper index denotes the dimension of the containing space). The point of intersection of these planes will be called the associated point for the point A_i, and the surface described by it—the associated surface for the net (A_i). Nets $L^{2n+1}_{n,n}$ exist with one arbitrary function of two arguments.

As in the case $N = 3$, the following holds.

Theorem. *If two consecutive nets* (A_i) *and* (A_{i+1}) *of the sequence* $\{A\}$ *belong to the class of nets* $L^{2n+1}_{n,n}$*, then all nets of this sequence belong to the same class.*

A Laplace sequence of nets $L^{2n+1}_{n,n}$ will be called a sequence $L^{2n+1}_{n,n}$. *Sequences* $L^{2n+1}_{n,n}$ *are determined by arbitrary* $8n - 2$ *functions of one argument.*

We shall indicate two characteristic criteria for such sequences:

1) the net of lines on one of the associated surfaces, corresponding to the focal net, must be conjugate;
2) the associated surfaces themselves must form a Laplace sequence $L^{2n+1}_{n,n}$ (an associated sequence).
3. By the axial plane of a conjugate net (A_i) of the even-dimensional projective space P_{2n} at the point A_i we shall mean the two-dimensional plane along which the $(n + 1)$-dimensional tangent planes of the lines of the net passing through the point A_i intersect; and by the axes of this net at the point A_i we shall mean the straight lines along which the $(n + 1)$-dimensional tangent plane of one of the lines of the net passing through the point A_i and the n-dimensional tangent plane of the other line intersect.

Thus, with every conjugate net of the space there are associated a two-parameter family of axial planes and two two-parameter families of axes.

By the focus of an axial plane we shall mean the point of intersection of this plane with an infinitely close axial plane. The axial plane of an arbitrary net (A_i) at the point A_i carries two foci, which are obtained as the intersections of this axial plane with infinitely close axial planes under displacements in the directions of the lines of the net. The focus $M_{i,i+1}$, obtained under displacement in the direction $A_{iA_{i-1}}$, will be called the point associated with the ray $A_{iA_{i+1}}$, and the focus $M_{i,i-1}$, obtained under displacement in the direction $A_{iA_{i+1}}$, the point associated with the ray $A_{iA_{i-1}}$. The points $M_{i,i+1}$ and $M_{i,i-1}$ will also be called associated with the point A_i. The surfaces described by the associated points will be called surfaces associated with the congruences $(A_{iA_{i+1}})$, $(A_{iA_{i-1}})$, or with the net (A_i).

The point $M_{i,i+1}$ can be obtained as the intersection of the axial planes of the nets (A_i) and (A_{i+1}) at the points A_i and A_{i+1}, as the intersection of the unlike axes of these nets at the same points, and, finally, as the intersection of the n-dimensional planes $[A_{i+1}A_{i+2} \dots A_{i+n+1}]$ and $[A_{iA_{i-1}} \dots A_{i-n}]$.

If on the surfaces $(M_{i,i+1})$ and $(M_{i,i-1})$ the nets corresponding to the net (A_i) are conjugate, then the net (A_i) will be called a net $L^{2n}_{n,n}$. As in the case $N = 4$, the following propositions are valid:

1) the net (A_i) belongs to the class of nets $L^{2n}_{n,n}$ if and only if the surfaces associated with it are related by a Laplace transformation;
2) if one of the nets of the sequence $\{A\}$ is of type $L^{2n}_{n,n}$, then all nets of this sequence are of the same type.

A Laplace sequence of nets $L^{2n}_{n,n}$ will be called a sequence $L^{2n}_{n,n}$. The arbitrariness of existence of such a sequence is equal to $8n-4$ functions of one argument. The surfaces associated with the nets of the sequence $L^{2n}_{n,n}$ themselves form a Laplace sequence (an associated sequence) of type $L^{2n}_{n,n}$.

4. Let us note a number of special classes of sequences $L^{N}_{n,n}$, characterized by a decrease in the dimensions of the planes whose intersection gives the associated points.

a) Sequences $L^{N}_{n-k,n-l}$ $(k, l = 1, \dots, n-2)$. They are characterized by the fact that the associated points serve as the intersection of planes of dimensions $n-k$ and $n-l$. The arbitrariness of existence of such sequences is equal to $8n-2-2(k+l)$ functions of one argument for $N = 2n+1$, and $8n-4-2(k+l)$ functions of one argument for $N = 2n$.

b) Sequences $L^{N}_{n-k,1}$ $(k = 0, 1, \dots, n-2)$. For them the associated points are obtained as a result of the intersection of a ray with the corresponding plane of dimension $n-k$. The arbitrariness of existence ⋯

is equal to $6n+1-2k$ $(N = 2n+1)$ and $6n-1-2k$ $(N = 2n)$ functions of one argument. The subclass of such sequences consists of sequences for which the indicated point of intersection serves as the focus of the ray of the sequence $\{A\}$ passing through it. The arbitrariness of existence in this case is reduced by one function of one argument.

c) **Sequences $L^{N}_{1,1}$.** The associated points are obtained as the intersection of two rays of the sequence $\{A\}$. The arbitrariness is equal to $2N+2$ functions of one argument. If the indicated point of intersection is the focus of one of the rays intersecting in it, then we obtain a subclass of sequences $L^{N}_{1,1}$, depending on $2N+1$ functions of one argument.

d) **Closed Laplace sequences.** If the point of intersection of two rays is the focus for both rays passing through it, then we have the most general closed $(N+1)$-term Laplace sequence, determined with arbitrariness $2N$ functions of one argument. Such a sequence is associated with itself.

For each of the indicated classes, the associated sequence is of the same type as the original one.

5. As has already been noted, the axial plane at the point A_0 of the conjugate net (A_0) of the space P_{2n}, generally speaking, has two foci—two associated A_0 points. But there exist nets having one or two focal lines.

For nets of the first kind, each line of one of the families of the conjugate net is immersed in its own $(n+1)$-dimensional tangent plane. Under displacement along a line of this family, the axial plane intersects the infinitely near one in a line passing through one of the associated points. The Laplace sequence in this case breaks off in one direction: the surface (A_n) is developable and has as its edge of regression the line into which the surface (A_{n+1}) degenerates. The arbitrariness of existence of nets of this type is equal to one function of two arguments.

For nets of the second kind, the indicated geometric characteristic occurs for both families. Such nets are determined with arbitrariness $4n + 2$ functions of one argument. The axial plane at each point of such a net has two focal lines: the first is obtained as the intersection of the axial plane with the infinitely near one under displacement along one line of the net, the second—along the other, while under all other displacements the point of intersection of these lines is obtained. This point of intersection describes the surface enveloped by the axial planes of the original net (A_0).

A special class of nets of this kind is formed by nets for which the focal lines merge into one line passing through the points $A_n = M_{0,-1}$, $A_n = M_{0,1}$ and remaining fixed under any displacements along the surface (A_0). The surfaces (A_n) and (A_{-n}) degenerate into this line. We have a special case of a closed $(2n+1)$-term Laplace sequence, two neighboring surfaces of which have degenerated into one and the same line. The net (A_0) generating such a sequence depends on $4n - 2$ functions of one argument.

6. Let a fixed hyperplane P_{N-1} be given in the space P_N. Denote by B_i the point of intersection of the ray $A_{iA_{i+1}}$ with this hyperplane. The Laplace sequence $\{A\}$ of the space P_N passes into the Laplace sequence

$$\dots, (B_{-n}), \dots, (B_{-1}),\ (B_0),\ (B_1), \dots, (B_n) \dots \tag{B}$$

of the space P_{N-1}.

If $N = 2n$, then the points of intersection of the axes of the nets (A_i) at the point A_i with the hyperplane P_{2n-1} lie on the axes of the nets (B_i) and (B_{i+1}), passing through the points B_i and B_{i+1}. If $N = 2n + 1$, then the points of intersection of the axes

of the nets (A_i) and (A_{i+1}) at the points A_i and A_{i+1} with the hyperplane P_{2n} lie on the axes of the net (B_i), passing through the point B_i.

Let $\{A\}$ be a sequence $L^N_{n,n}$. Requiring that the sequence $\{B\}$ of the space P_{N-1} be a sequence $L^{N-1}_{n-1,n-1}$, we obtain that this is possible if and only if the sequence $\{A\}$ in P_N is a sequence $L^N_{n,n}$ such that the sequence associated with it is immersed in P_{N-1}. In this case, for $N = 2n$ the sequence $\{B\}$ in P_{2n-1} turns out to be the most general sequence $L^{2n-1}_{n-1,n-1}$, while for $N = 2n + 1$ the sequence $\{B\}$ in P_{2n} is of type $L^{2n}_{n-1,n-1}$, i.e., it turns out to be the most general sequence of one of the special types. The arbitrariness of the existence of such sequences $\{A\}$ in P_N is equal to $8n-6$ functions of one argument for $N=2n$, and $8n-4$ functions of one argument for $N=2n+1$.

Under such a mapping, the special classes of sequences $N^N_{n,n}$ pass into special ones:

a) the sequence $L^N_{n-k,n-l}$ passes into the sequence $L^{N-1}_{n-k-1,n-l-1}$ (the arbitrariness of such sequences $L^N_{n-k,n-l}$ is equal to $8n-6-2(k+l)$ functions of one argument for $N=2n$, and $8n-4-2(k+l)$ functions of one argument for $N=2n+1$);

b) the sequence $L^N_{n-k,1}$ passes into the sequence $L^{N-1}_{n-k-1,1}$, and the point of intersection of the ray with the corresponding plane is the focus of the ray (the arbitrariness is equal to $6n-2k-3$ functions of one argument for $N=2n$, and $6n-2k-1$ functions of one argument for $N=2n+1$);

c) the sequence $L^N_{1,1}$ passes into the most general N-term closed Laplace sequence of the space P_{N-1} (the arbitrariness is equal to $2N$ functions of one argument).

In all the cases indicated, the sequences $\{A\}$ and $\{B\}$ have common associated sequences.

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Note: Figure translations are in progress. See original paper for figures.

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