CLASSES OF DOMAINS AND EMBEDDING THEOREMS FOR FUNCTIONAL SPACES

1960

SovietRxiv

Abstract

Full Text

MATHEMATICS

V. G. Maz' ya

CLASSES OF DOMAINS AND EMBEDDING THEOREMS FOR FUNCTIONAL SPACES

(Presented by Academician V. I. Smirnov on 28 III 1960)

The embedding theorems of S. L. Sobolev (1) were proved for functions defined in domains that can be divided into a finite number of domains star-shaped with respect to a ball. The proof made essential use of S. L. Sobolev' s integral representation and the properties of potential-type operators. Embedding theorems are the subject of works by V. P. Il' in (2), who, using an analogous method, obtained new results. V. P. Il' in considered functions defined in domains satisfying the "cone condition." In (3) E. Gagliardo gave a new proof of embedding theorems for domains satisfying the "cone condition," and obtained some new results.

In the present note we formulate sufficient, and in some cases necessary and sufficient, conditions on a domain under which various embedding theorems are valid. The method of proof differs from the methods of S. L. Sobolev and E. Gagliardo.

Let D be a domain of the n-dimensional Euclidean space E_n. By the space $L_p^{(i)}(D)$ we shall mean the closure of $C^{(i)}(\overline{D})$ in the norm

$$\|u\|_{L_p^{(i)}(D)} = \|\operatorname{grad}_i u\|_{L_p(D)},$$

where

$$\operatorname{grad}_i u = \left\{ \frac{\partial^i u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}} \right\}, \qquad \sum_\nu \alpha_\nu = i, \quad i \geqslant 0.$$

By the space $W_p^{(l)}(D)$ $(p \geqslant 1)$ we shall mean the closure of $C^{(l)}(\overline{D})$ in the norm

$$\|u\|_{W_p^{(l)}(D)} = \sum_{i=0}^{l} \|u\|_{L_p^{(i)}(D)}.$$

In what follows the boundary of an arbitrary set E will be denoted by ΓE.

Consider a function $u(x) \in C^{(1)}(\overline{D})$. For simplicity assume that $u > 0$ and $|\operatorname{grad} u| > 0$ everywhere except, perhaps, on a set of n-dimensional measure zero. Construct the level surfaces of the function $u(x)$. These surfaces generate a continuum of nested subsets E_μ, where $\mu = \operatorname{mes}_n E_\mu$, such that $u(x) \geqslant u(y)$

if $x \in E_\mu,\ y \in D \setminus E_\mu$. The level surface corresponding to the set E_μ will be denoted by S_μ. Suppose also that $\operatorname{mes}_n E\{x;\ u(x) \geqslant t\} < \infty$ for $t \in (0, \infty)$.

Definition 1. A domain D belongs to the class I^n_{p,q^*} $(p > 1,\ q^* \geqslant 1)$ if there exist positive constants $M \leqslant \operatorname{mes}_n D$ and $\mathfrak{A}(M) < \infty$ such that for every family of level surfaces S_μ $(0 \leqslant \mu \leqslant M)$ the inequality

$$\mu^{p'/q^*} \int_\mu^M \varphi_p^{-1}(\tau)\, d\tau \leqslant \mathfrak{A}(M), \tag{1}$$

holds, where

$$\varphi_p^{p-1}(\tau) = \int_S \left(\frac{d\tau}{d\nu}\right)^{p-1} ds;$$

$d\nu$ is the distance along the normal to S_τ between S_τ and $S_{\tau+d\tau}$, and

$$p^{-1} + {p'}^{-1} = 1.$$

Definition 2. A domain D belongs to the class $J^{(n)}_\alpha$ $\left(\alpha \geq \frac{n-1}{n}\right)$ if there exist positive constants $M \leq \operatorname{mes}_n D$ and $\mathfrak{B}(M) < \infty$, depending only on the domain D, such that for $\mu \leq M$ the inequality

$$\mu^\alpha \leq \mathfrak{B}(M) \operatorname{mes}_{n-1} S_\mu. \tag{2}$$

holds.

Example 1. A domain satisfying the "cone condition" (^{2},^{3}) belongs to the class $J^{(n)}_{\frac{n-1}{n}}$, but, for example, the domain $D \in J^{(2)}_{1/2}$, composed of the squares

$$\{2^{-k} \leq x \leq 2^{-k+1},\ 0 < y \leq 2^{-k}\} \quad (k = 0, 1, \ldots),$$

does not satisfy the "cone condition."

Example 2. The n-dimensional cone

$$\left\{ \left(\sum_{i=1}^{n-1} x_i^2 \right)^{1/2} < x_n^\beta,\ 0 < x_n < 1 \right\} \quad (\beta \geq 1)$$

belongs to the class $J^{(n)}_{\beta(n-1)/\beta(n-1)+1}$.

Theorem 1. If $p \leq q^*$, $q \leq q^*$, or $p > q^*$, $q < q^*$, then, for the validity of the inequality*

$$\|u\|_{L_q(D)} \leq c \left\{ A_p \| \operatorname{grad} u \|_{L_p(D)} + M^{\frac{s-q^*}{sq^*}} \|u\|_{L_s(D)} \right\}^{\frac{q^*(q-r)}{q(q^*-r)}} \|u\|_{L_r(D)}^{\frac{r(q^*-q)}{q(q^*-r)}}, \tag{3}$$

where $s < q^*$, $r < q$, $p \geq 1$, it is necessary and sufficient that the domain D belong to the class $I_{p,q^*}^{(n)}$ for $p > 1$ ($A_p = \mathfrak{A}^{1/p'}$) and to the class $J_{1/q^*}^{(n)}$ for $p = 1$ ($A_1 = \mathfrak{B}$).

In the case $p > q^*$, $q = q^*$, the theorem is false.

One can prove the embedding

$$J_{1,q^*+1/p'}^{(n)} \subset I_{p,q^*}^{(n)} \quad (p > 1).$$

A consequence of this is:

Theorem 2. If $D \in J_\alpha^{(n)}$, $p \geq 1$, $p(1-\alpha) < 1$,

$$q^* = \frac{p}{1 + p(\alpha - 1)},$$

and if $q \leq q^*$ for $\frac{n-1}{n} \leq \alpha \leq 1$, or $q < q^*$ for $\alpha > 1$, then inequality (3) holds ($A_p = \mathfrak{B}$).

If $p > 1$, $\alpha < 1$, $p(1-\alpha) = 1$, then for any positive q inequality (3) holds ($A_p = \mathfrak{B}$).

If $\alpha > 1$, then for the limiting exponent $q = q^*$ the theorem is false.

Remark. In Theorem 2, for $p > 1$ and $q < q^*$, condition (2) can be weakened by replacing it, for example, by the inequality

$$\mu^\alpha (1 + |\ln \mu|^\delta)^{-1} \leq \mathfrak{B}'(M) \operatorname{mes}_{n-1} S_\mu, \quad (\delta > 0).$$

Definition 3. A domain D belongs to the class $\tilde{I}_{p,q^*}^{(n)}$ ($p > 1$, $q^* \geq 1$), if $D \in I_{p,q^*}^{(n)}$ and $\inf \mathfrak{A}(M) \to 0$ as $M \to 0$.

Definition 4. A domain D belongs to the class $\widetilde{J}_\alpha^{(n)}$ $\left(\alpha > \frac{n-1}{n}\right)$, if $D \in J_\alpha^{(n)}$ and $\inf \mathfrak{B}(M) \to 0$ as $M \to 0$.

We now indicate conditions for complete continuity of the embedding operator.

Theorem 3. Let $\operatorname{mes}_n D < \infty$.

1) For complete continuity of the embedding operator $L_p^{(1)}(D) \cap L_s(D)$ into $L_{q^*}(D)$ ($p \leq q^*$), it is necessary and sufficient that for $p > 1$, $D \in \tilde{I}_{p,q^*}^{(n)}$, and for $p = 1$, $D \in \widetilde{J}_{1/q^*}^{(n)}$.
2) Under the assumptions of Theorem 1, for $p > q^* > q$, the embedding operator $L_p^{(1)}(D) \cap L_s(D)$ into $L_q(D)$ is completely continuous.

---

* Here and below c denotes constants depending only on n, p, q, q^*, r, s.

From the formulated results it follows:

Theorem 4. If $\operatorname{mes}_n D < \infty$, $D \in J_\alpha^{(n)}$ $\left(\alpha \geqslant \dfrac{n-1}{n}\right)$ and if:

a) $p(1-\alpha) = 1,\ p > 1,\ q \geqslant 1$ is an arbitrary number, or

b) $p(1-\alpha) < 1,\ p \geqslant 1,\ q < q^* = \dfrac{p}{1+p(\alpha-1)}$, then the embedding operator $L_p^{(1)}(D) \cap L_s(D)$ into $L_q(D)$ is completely continuous.

In the limiting case $q = q^*$ the theorem is false.

Remark. The Poincaré inequality

$$\|u\|_{L_2(D)} \leqslant K\{\|\operatorname{grad} u\|_{L_2(D)} + \|u\|_{L(D)}\},$$

as is known, is valid for domains of type $\mathfrak{R}$ (4). It can be shown that every domain of type $\mathfrak{R}$ belongs to the class $J_1^{(2)}$, but, for example, the domain $D \in J_1^{(2)}$ equal to the sum of the squares $Q_n\{2^{-n} < x < 3 \cdot 2^{-n-1},\ 0 < y < 2^{-n-1}\}$ and the rectangles $P_n : \{3 \cdot 2^{-n-2} \leqslant x \leqslant 2^{-n},\ 0 < y < 1\}$ $(n = 0, 1 \ldots)$, is not a finite sum of normal domains.

Consider the question of the embedding of $L_p^{(1)}(D) \cap L_s(D)$ into $C(D)$, where $C(D)$ is the space of functions continuous and bounded in $\overline{D}$.

Theorem 5. If $D \in J_\alpha^{(n)}$, where $1 > \alpha \geqslant \dfrac{n-1}{n}$, and if $p(1-\alpha) > 1,\ r > 0,\ s > 0$, then the inequality holds

$$\|u\|_{C(D)} \leqslant c\left\{\mathfrak{B}(M)\|\operatorname{grad} u\|_{L_p(D)} + M^{\frac{1-p(1-\alpha)}{p} - \frac{1}{s}}\|u\|_{L_s(D)}\right\}^{\frac{p}{r[p(1-\alpha)-1]+p}} \|u\|_{L_r(D)}^{\frac{r[p(1-\alpha)-1]}{r[p(1-\alpha)-1]+p}} \tag{4}$$

If $\operatorname{mes}_n D < \infty$, then the embedding operator $L_p^{(1)}(D) \cap L_s(D)$ into $C(D)$ is completely continuous.

In the limiting case $p(1-\alpha) = 1$ the theorem is false.

Corollary. If $\operatorname{mes}_n D < \infty$ and $D \in J_\alpha^{(n)}$ $\left(1 > \alpha \geqslant \dfrac{n-1}{n}\right)$, and if $lp(1-\alpha) > 1$, then the embedding operator $W_p^{(l)}(D)$ into $C(D)$ is bounded and completely continuous.

There is also a similar corollary of Theorems 2 and 4.

One may also consider other classes of domains, analogous to $J_{p,q^*}^{(n)}$ and $J_{\alpha}^{(n)}$, and prove embedding theorems for them. Define, for example, a class of domains satisfying the following condition. There exist constants M and $\mathfrak{F} < \infty$, depending only on the domain, such that for $\mu \geqslant M$ the inequality

$$\mu^{\alpha} \leqslant \mathfrak{F} \operatorname{mes}_{n-1} S_{\mu}, \tag{5}$$

holds, where $0 < \alpha \leqslant \dfrac{n-1}{n}$. Condition (5) is satisfied, for example, by the two-dimensional domain

$$\{0 < x < \infty, \quad 0 < y < x^{\frac{\alpha}{1-\alpha}}\} \quad (0 < \alpha \leqslant 1/2).$$

Consider the conditions under which the space $L_p^{(1)}(D) \cap L_s(\Gamma D)$ is embedded in $L_q(D)$. Suppose that the boundary of the domain D is measurable (in the Hausdorff sense).

Definition 5. A domain D belongs to the class $K_{\beta}^{(n)}$ $(\beta > 0)$ if there exist two constants $\mathfrak{C}$ and $\mathfrak{D}$, depending only on the domain D, such that

$$\mu^{\frac{n-1}{n}} \leqslant \mathfrak{C} \operatorname{mes}_{n-1} S_{\mu} + \mathfrak{D} \operatorname{mes}_{n-1}^{\beta}(\Gamma E_{\mu} \setminus S_{\mu}). \tag{6}$$

Example 3. The two-dimensional domains $D_1 : \{0 < x < 1,\ 0 < y < x^{2\beta-1}\}$ $(\beta > 1)$ and $D_2 : \{0 < x < \infty,\ 0 < y < (x+1)^{2\beta-1}\}$ $(0 < \beta < 1)$ belong to the class $K_{\beta}^{(2)}$.

Let us formulate the theorem for the limiting exponent $q^* = \dfrac{pn}{n-p}$.

Theorem 6. 1) If $D \in K_{\beta}^{(n)}$, where $\beta \geq 1$, then the inequality

$$\|u\|_{L_{\frac{pn}{n-p}}(D)} \leq c\left\{\mathfrak{C}\|\operatorname{grad} u\|_{L_p(D)} + \mathfrak{D}\|u\|_{L_{\frac{p(n-1)}{(n-p)\beta}}(\Gamma D)}\right\}; \tag{7}$$

holds;

2) if $D \in K_{\beta}^{(n)}$, where $\beta < 1$, then the inequality

$$\|u\|_{L_{\frac{pn}{n-p}}(D)} \leq c\left\{\mathfrak{C}\|\operatorname{grad} u\|_{L_p(D)} + \mathfrak{D}\|u\|_{L_r^{\rho}(\Gamma D)}\|u\|_{L_s^{\sigma}(\Gamma D)}\right\}, \tag{8}$$

holds,

where $\rho = r[p(n-1) - \beta s(n-p)]/p(r-s)(n-1)$, $\sigma = s[\beta r(n-p) - p(n-1)]/p(r-s)(n-1)$, $s < p(n-1)/\beta(n-p) < r$.

For $p = 1$, the condition $D \in K_{\beta}^{(n)}$ is also necessary.

In the case $\beta < \dfrac{n-1}{n}$, for $s = p(n-1)/\beta(n-p)$ the theorem is false.

If one assumes only that the boundary D is measurable, then $D \in K_1^{(n)}$, since condition (6) is the classical isoperimetric inequality. In this case inequality (7) for $p = 1$ has the form

$$\|u\|_{L_{\frac{n}{n-1}}(D)} \leq \frac{1}{n\sqrt{\pi}} \Gamma^{\frac{1}{n}} \left(1 + \frac{n}{2}\right) \left(\|\operatorname{grad} u\|_{L(D)} + \|u\|_{L(\Gamma D)}\right).$$

The constant is sharp.

If $D \in J_\alpha^{(n)} \cap K_\beta^{(n)}$, $\operatorname{mes}_n D < \infty$, then there exist constants $\mathfrak{B}'$ and $\mathfrak{D}'$ such that

$$\mu^\gamma \leq \mathfrak{B}' \operatorname{mes}_{n-1} S_\mu + \mathfrak{D}' \operatorname{mes}_{n-1}^{\beta\gamma n/(n-1)} (\Gamma E_\mu \setminus S_\mu) \quad \text{for } \frac{n-1}{n} \leq \gamma < \alpha. \tag{9}$$

Starting from condition (9), one can prove inequalities analogous to (7), (8). We restrict ourselves to the case $\dfrac{n-1}{n} \leq \gamma < \alpha \leq 1$, $\beta\gamma n \geq n-1$. Then for $p \geq 1$ and $p(1-\gamma) < 1$ the inequality holds

$$\|u\|_{L_{\frac{\gamma p}{1+(\gamma-1)p}}(D)} \leq c \left\{ \mathfrak{B}' \|\operatorname{grad} u\|_{L_p(D)} + \mathfrak{D}' \|u\|_{L_{\frac{p(n-1)}{[p(\gamma-1)+1]\beta n}}(\Gamma D)} \right\}.$$

In conclusion, I express my sincere gratitude to Prof. S. G. Mikhlin for his attention to the present work.

Leningrad State University
named after A. A. Zhdanov

Received
17 III 1960

REFERENCES

1. S. L. Sobolev, *Some applications of functional analysis in mathematical physics*, L., 1950.
2. V. P. Il' in, *Tr. Mat. Inst. im. V. A. Steklova AN SSSR*, **53**, p. 64 (1959).
3. E. Gagliardo, *Ricerche di mat.*, **7**, Fasc. I° (1958).
4. R. Courant, D. Hilbert, *Methods of Mathematical Physics*, 2, 1951.

Note: Figure translations are in progress. See original paper for figures.