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Abstract

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ON THE CONNECTION BETWEEN THE GROWTH OF A MEROMORPHIC FUNCTION AND THE DISTRIBUTION OF ITS VALUES BY ARGUMENTS

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By investigations of B. Ya. Levin and A. Pfluger (for the main results, as well as references to the original papers, see (^{1})) it was established that certain regularity requirements imposed on the **moduli** of the zeros of an entire function entail regularity of its growth. The purpose of the present note is to show that certain restrictions imposed only on the **arguments** of the zeros and ones of an entire function entail a rather strong regularity of its growth*. We shall show here that these results carry over also to meromorphic functions.

1°. We shall adhere to the following notation: $f(z)$ is a function meromorphic in the whole finite plane; α and β are numbers satisfying the inequality $0 \le \alpha < \beta \le 2\pi$, $\gamma = \beta - \alpha$; $\{\alpha_j\}_{j=1}^{n}$ are numbers satisfying the inequality $0 \le \alpha_1 < \alpha_2 < \dots < \alpha_n < 2\pi$; (R) is the system of rays $\arg z = \alpha_j$; $\gamma_j = \alpha_{j+1} - \alpha_j$ $(\alpha_{n+1} = \alpha_1 + 2\pi)$; $\theta = \min_{1\le j\le n} \gamma_j$, $\Theta = \max_{1\le j\le n} \gamma_j$; $O(1)$ is a quantity remaining bounded as $r \to \infty$; $\{a_k\}$ are the poles, $\{b_l\}$ the zeros, $\{c_m\}$ the ones of the function $f(z)$, lying outside the disk $|z| < 1$, considered with multiplicities; $A_{\alpha\beta}(r, f)$, $B_{\alpha\beta}(r, f)$, $C_{\alpha\beta}(r, f)$, $S_{\alpha\beta}(r, f) = A_{\alpha\beta}(r, f) + B_{\alpha\beta}(r, f) + C_{\alpha\beta}(r, f)$ are the quantities (everywhere $r \ge 1$) introduced by Nevanlinna** (^{3}), characterizing the distribution of the values of the function $f(z)$ in the angle $\alpha < \arg z < \beta$. Recall that the quantity $C_{\alpha\beta}(r, f)$ characterizes the distribution of the poles of $f(z)$ in this angle, taking their arguments into account in an essential way. In particular, the relation $C_{\alpha\beta}(r, f) = O(1)$ is equivalent to the convergence of the series

$$\sum_{\alpha \le \varphi_k \le \beta} \left[\sin \frac{\pi}{\gamma}(\varphi_k - \alpha)\right] r_k^{-\pi/\gamma} \qquad (a_k = r_k e^{i\varphi_k}).$$

We shall use the following results.

Theorem A. If the function $f(z)$ satisfies the condition

$$\int_1^\infty \ln^+ T(r,f)\, r^{-\pi/\gamma-1}\, dr < \infty,$$

then for any finite set of $q \geq 3$ distinct complex numbers $a^{(1)}, a^{(2)}, \dots, a^{(q)}$ from the extended plane the following relation holds***

$$(q-2)S_{\alpha\beta}(r,f) \leq \sum_{\nu=1}^{q} C_{\alpha\beta}(r,(f-a^{(\nu)})^{-1}) + O(1).$$

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* We note that a qualitative assertion close to this was formulated in the paper of A. A. Goldberg (^{2}).

** Definitions of these quantities can also be found in (^{9}).

*** By definition, for $a = \infty$, $C_{\alpha\beta}(r,(f-a)^{-1}) = C_{\alpha\beta}(r,f)$.

Theorem B. If $S_{0\pi}(r,f) = O(1)$, then there exists a finite limit

$$\eta = \lim_{r\to\infty} r^{-1} \int_0^\pi \ln|f(re^{i\vartheta})| \sin\vartheta\, d\vartheta$$

and the representation holds

$$\ln|f(z)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{r\sin\varphi \ln|f(t)|}{r^2+t^2-2rt\cos\varphi}\, dt + \frac{2\eta}{\pi} r\sin\varphi +$$
$$+ \sum_{\operatorname{Im} a_k>0} \ln\left|\frac{z-\overline{a}_k}{z-a_k}\right| - \sum_{\operatorname{Im} b_l>0} \ln\left|\frac{z-\overline{b}_l}{z-b_l}\right| \qquad (z = re^{i\varphi}). \tag{1}$$

Theorems A and B are due to Nevanlinna (^{3}), who in Theorem A assumed, instead of our condition, the finite order of $f(z)$. In order to obtain the result in the form in which we have formulated it, one must use the estimate

$$A_{\alpha\beta}(r, f'f^{-1}) = O\left(\int_1^{2r} \ln^+ T(r,f) r^{-\pi/\gamma-1}\, dr\right).$$

Let the function $u(r,\varphi)$ be defined for $1 \leq r < \infty$, $\alpha \leq \varphi \leq \beta$. If it is possible to specify a set $E \subset [1,\infty)$ of finite logarithmic length $\left(\text{i.e. } \int_E d\ln r < \infty\right)$ such that $\lim_{\substack{r\to\infty\\ r\notin E}} u(r,\varphi) = h(\varphi)$ exists uniformly in φ, $\alpha \leq \varphi \leq \beta$, then we shall agree to say that for $\alpha \leq \varphi \leq \beta$ there exists

$$\overset{(l)}{\lim_{r\to\infty}}\, u(r,\varphi) = h(\varphi).$$

Theorem V. If the function $f(z)$ is representable in the form (1), then for $0 \le \varphi \le \pi$ there exists

$$\overset{(l)}{\lim_{r\to\infty}} r^{-1} \ln |f(re^{i\varphi})| = 2\eta\pi^{-1} \sin\varphi.$$

This theorem is a simple consequence of a result of Heiman (^{4}). We note that Theorems B and V are easily generalized to the case of an angle of arbitrary opening.

Theorem G. If the function $f(z)$ satisfies the conditions:

a) for at least two distinct values a from the extended plane and for some system of rays (R)

$$\sum_{j=1}^{n} C_{\alpha_j \alpha_{j+1}}(r, (f-a)^{-1}) = O(1);$$

b) for some value a, different from those which occur in condition a), the positive quantity

$$\Delta^*(a) = \sup_{\mathfrak{B}\in K} \lim_{\substack{r\to\infty \\ r\in\mathfrak{B}}} \frac{m(r,a)}{T(r,f)}$$

(here K denotes the class of sets lying on the positive half-axis with upper density* less than 1),

then the growth of $f(z)$ is not higher than order $\pi\delta^{-1}$ and of normal type.

This theorem is a special case of the main result of the note (^{5}).

2°. From a comparison of Theorems A, B, and V the following follows immediately:

Theorem 1. If the function $f(z)$ satisfies the conditions:

1) there exist α and β such that, for at least three distinct values a from the extended plane,

$$C_{\alpha\beta}(r, (f-a)^{-1}) = O(1);$$

2)

$$\int_1^\infty \ln^+ T(r,f) r^{-\pi/\gamma - 1}\, dr < \infty,$$

[*] The upper density of a set $E \subset [1, \infty)$ is the quantity

$$\lim_{r\to\infty} r^{-1}\,\mathrm{mes}\{E \cap [1, r]\}.$$

then

3) for $\alpha \leqslant \varphi \leqslant \beta$ there exists

$$\overset{(l)}{\lim_{r\to\infty}}\, r^{-\pi/\gamma} \ln|f(re^{i\varphi})| = c \sin\frac{\pi}{\gamma}(\varphi - \alpha);$$

4) the integrals

$$\int_1^\infty |\ln|f(te^{i\alpha})||\, t^{-\pi/\gamma-1}\, dt \quad \text{and} \quad \int_1^\infty |\ln|f(te^{i\beta})||\, t^{-\pi/\gamma-1}\, dt$$

converge.

Corollary 1. Let there be a certain system of rays (R). If the function $f(z)$ satisfies the conditions:

1) for at least three distinct values a from the extended plane

$$C_{\alpha_j\alpha_{j+1}}(r, (f-a)^{-1}) = O(1) \quad (j = 1, 2, \dots, n);$$

2)

$$\int_1^\infty \ln^+ T(r, f) r^{-\pi/\Theta-1}\, dr < \infty,$$

then

3) for $\alpha_j \leqslant \varphi \leqslant \alpha_{j+1}$ there exists

$$\overset{(l)}{\lim_{r\to\infty}}\, r^{-\pi/\gamma_j} \ln|f(re^{i\varphi})| = c_j \sin\frac{\pi}{\gamma_j}(\varphi - \alpha_j)$$

$$(j = 1, 2, \dots, n);$$

4) the integrals

$$\int_1^\infty |\ln|f(te^{i\alpha_j})||\, t^{-\pi/\delta_j-1}\, dt \quad (\delta_j = \max(\gamma_j, \gamma_{j-1}),$$

$$\gamma_0 = \gamma_n,\ j = 1, \dots, n)$$

converge.

Condition 1) can be weakened somewhat, replacing it by the requirement that for each j there is such a triple of numbers a from the extended plane that

$$C_{\alpha_j\alpha_{j+1}}(r,(f-a)^{-1}) = O(1).$$

3°. In this item we shall consider functions having the representation

$$f(z) = \sum_{k=1}^{\infty} \frac{A_k}{z-a_k}, \qquad \sum_{a_k \neq 0} \left| \frac{A_k}{a_k} \right| < \infty. \tag{2}$$

M. V. Keldysh proved $(^6)$ that for such functions

$$m(r,f) = O(1), \tag{3}$$

whence it follows easily that for any α and β one has

$$B_{\alpha\beta}(r,f) = O(1).$$

By a method close to that which was used in $(^6)$ to prove (3), one can establish that for functions of the form (2) one has

$$A_{\alpha\beta}(r,f) = O(1),$$

if $0 < \gamma \leqslant \pi$. Consequently, for functions of the form (2) always

$$S_{\alpha\beta}(r,f) = C_{\alpha\beta}(r,f) + O(1) \quad (0 < \gamma \leqslant \pi).$$

From this relation and Theorems B and C it follows:

Theorem 2. If the function $f(z)$ is represented in the form (2) and, for some α and β, $0 < \gamma \leqslant \pi$,

$$C_{\alpha\beta}(r,f) = O(1),$$

then for $f(z)$ the assertions of Theorem 1 hold; moreover, in 3) we shall have $c \leqslant 0$.

Corollary 2. If the function $f(z)$ is represented in the form (2) and for some system of rays (R) with $\Theta \leqslant \pi$ one has

$$\sum_{j=1}^{n} C_{\alpha_j\alpha_{j+1}}(r,f) = O(1),$$

then for $f(z)$ the assertions of Corollary 1 hold, and in 3) we shall have

$$c_j \leqslant 0 \quad (j = 1, 2, \dots, n).$$

This result is a generalization of a theorem of M. G. Krein $(^7)$.

4°. From Theorem C it follows that condition 2) of Corollary 1 can be replaced by the following condition:

2^{**}) There is at least one value a (not necessarily distinct from those which occur in condition 1)) for which the quantity $\Delta^*(a)$ is positive.

Since for every entire function $\Delta^*(\infty) = 1$, it follows from this that

Theorem 3. If $f(z)$ is an entire function such that for at least two distinct values $a \neq \infty$

$$\sum_{j=1}^{n} C_{\alpha_j \alpha_{j+1}} \left(r, (f - a^{-1}) = O(1)\right),$$

then the assertions of Corollary 1 are valid for this function.

Corollary 3. If $f(z)$ is an entire function for which $\sum |\operatorname{Im}(b_l^{-1})| < \infty$, $\sum |\operatorname{Im}(c_m^{-1})| < \infty$, then $f(z)$ is a function of exponential type satisfying the condition

$$\int_{-\infty}^{\infty} \frac{|\ln |f(t)||}{1+t^2}\, dt < \infty.$$

5°. Using some results of Mii[8], one can somewhat strengthen the results of 2°, 3°, and 4°, by considering, in addition to the a-points of the function, also the a-points of its derivatives. In doing so it is also necessary to use some results from [5].

6°. We regard as highly probable the supposition that condition 2) in Theorem 1 is superfluous.

It seems to us that it would be interesting to prove or refute the possibility of replacing, in Corollary 3, the condition $\sum |\operatorname{Im}(c_m^{-1})| < \infty$ by the condition $\delta(1) > 0$ or by the condition $\Delta^*(1) > 0$. We note that in [5] it was established (in particular) that if $\sum |\operatorname{Im}(b_l^{-1})| < \infty$ and $\Delta^*(1) > 0$, then $f(z)$ is of exponential type.

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Note: Figure translations are in progress. See original paper for figures.

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