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Abstract

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TEMPERATURE EFFECTS IN PLASMA OSCILLATIONS OF A HIGH-DENSITY FERMI GAS

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In the present paper the spectrum of collective oscillations of a high-density electron gas is investigated. The method used makes it possible to treat the cases of high and low temperatures in a unified way. The only difference arises in evaluating the integral entering into the final expression for the energy of the collective oscillations. The value of this integral can be calculated by expanding the integrand in a series. For low temperatures one obtains a temperature correction that vanishes at temperature equal to zero. At high temperatures the correction is of a quantum character and disappears in the classical limit.

The study of the spectrum of elementary excitations of an electron gas is now naturally carried out within the framework of the method of Gell-Mann and Brueckner ([1]), by which it was first possible to sum all the diagrams most important in the high-density approximation for the case of temperature equal to zero. Such diagrams include all those in which the particle–hole complex remains unbroken and in which this complex is the only basic element. In the present work, instead of the original formulation of the Gell-Mann–Brueckner method, a variant of the method of approximate second quantization is used ([2,3]). A model Hamiltonian is constructed in such a way that the particle–hole complex remains unbroken. The vertex parts of the model Hamiltonian are constructed under the condition that they give the same contribution as the vertex parts of the exact Hamiltonian which do not break the particle–hole complex. H_0^M is chosen so as to ensure the correct energy denominators in the perturbation-theory series. It is not difficult to see that the indicated method is equivalent to summing the Gell-Mann–Brueckner diagrams.

On the exact Hamiltonian we perform the transformation introduced in the work of Chen Chun-hsien ([4]):

$$a_k^+ \to (1 - n_k)^{1/2} c_k^+ + n_k^{1/2} b_k,$$

$$a_k \to (1-n_k)^{1/2} c_k + n_k^{1/2} b_k^+, \tag{1}$$

where

$$n_k = \left(\exp \frac{E(k)-\mu}{\theta} + 1 \right)^{-1}$$

is the Fermi distribution.

In its idea, this method goes back to the work of Bloch and Dominicis ([5]), who succeeded in generalizing Wick' s theorem to the temperature case. However, transformation (1) is not completely equivalent to the method of Bloch and Dominicis. In our case it makes it possible to obtain the result by an easier route than direct summation of diagrams.

The exact Hamiltonian in the second-quantization representation has the form

$$H = H_0 + H_{\text{int}} = \sum_k E(k) a_k^+ a_k + \sum_{q\neq 0} \frac{\nu(q)}{2\Omega} \sum_{kk'} a_{k+q}^+ a_{k'-q}^+ a_{k'} a_k, \tag{2}$$

where Ω is the volume of the system, q is the momentum, k is the totality of momentum and spin, $E(k) = k^2/2m$, $\nu(q) = 4\pi e^2 \hbar^2/q^2$.

Substituting transformation (1) into (2), we select the terms, using the considerations given above. For the particle–hole complex operators we introduce the notation

$$\beta_q^+(k) = c_{k+q}^+ b_k^+; \qquad \beta_q(k) = b_k c_{k+q}. \tag{3}$$

The final form of the model Hamiltonian is the following:

$$\begin{aligned} H^M = & \sum \{E(p+q) - E(p)\}\, \beta_q^+(p) \beta_q(p) + \\ & + \frac{1}{\Omega} \sum_{q\neq 0} \nu(q) \sum_{k,k'} n_k^{1/2} n_{k'}^{1/2} (1-n_{k+p})^{1/2} (1-n_{k'+1})^{1/2} \beta_q^+(k) \beta_q(k') + \\ & + \frac{1}{2\Omega} \sum_{q\neq 0} \nu(q) \sum_{k,k'} n_k^{1/2} n_{k'}^{1/2} (1-n_{k'-q})^{1/2} (1-n_{k+q})^{1/2} \beta_q^+(k) \beta_{-q}^+(k') + \\ & + \frac{1}{2\Omega} \sum_{q\neq 0} \nu(q) \sum_{k,k'} n_k^{1/2} n_{k'}^{1/2} (1-n_{k'-q})^{1/2} (1-n_{k+q})^{1/2} \beta_q(k) \beta_{-q}(k'). \end{aligned} \tag{4}$$

Its diagonalization leads to the system of equations ([6])

$$Eu_q(k) = \{E(p+q) - E(p)\}u_q(k) +$$
$$+ \frac{\nu(q)}{\Omega}(1-n_{k+q})^{1/2} n_k^{1/2} \sum_{k'} n_{k'}^{1/2}\{(1-n_{k'-q})^{1/2} v_{-q}(k') + (1-n_{k'+q})^{1/2} u_q(k')\},$$
$$-Ev_q(k) = \{E(p+q) - E(p)\}v_q(k) + \qquad (5)$$

$$+\frac{\nu(q)}{\Omega}(1-n_{k+q})^{1/2} n_k^{1/2} \sum_{k'} n_{k'}^{1/2}\{(1-n_{k'-q})^{1/2} u_{-q}(k') + (1-n_{k'+q})^{1/2} v_q(k')\}$$

with the normalization condition

$$\sum_{1k}\{u_q^2(k) - v_q^2(k)\} = 1. \qquad (6)$$

The eigenvalues of system (5) are the eigenvalues of Hamiltonian (4).

From (5) we obtain the dispersion relation for the energy

$$1 = \frac{4\nu(q)}{\Omega} \sum_p n_p(1-n_{p+q}) \frac{E(p+q) - E(p)}{E^2 - \{E(p+q) - E(p)\}^2}. \qquad (7)$$

In the case of zero temperature, (7) was obtained in the works of Ferrell ([7]) and Sawada ([8]). A relation equivalent to (7) for nonzero temperature was obtained in the works of Klimontovich and Silin ([16]) and Larkin ([17]). At high temperatures and $\hbar \to 0$, (7) coincides with the known plasma relations ([9,10]).

It is known that (7) has two types of solutions ([7,8]). In the present work we are interested in the isolated root, since it characterizes the collective properties of the system. The second type of solutions of (7) corresponds to individual excitations.

To find the isolated root, we pass in (7) from the sum to an integral

$$1 = \frac{4\nu(q)}{(2\pi\hbar)^3} \int d^3p\, n_p(1-n_{p+q}) \frac{pq/m + q^2/2m}{E^2 - (pq/m + q^2/2m)^2}. \qquad (8)$$

For the case $T = 0$, the value of integral (8) is given in ([11]), but at a nonzero temperature the calculation is made difficult by the presence of the function n_p. However, when considering the case important for us of small q, i.e., long waves, we can dispense with this calculation. For the collective branch

the indicated region of small q is the most essential, since for large q collective oscillations are not observed at all. Moreover, the approximation method we have used is the better the longer the wavelength, i.e., the smaller q is. Therefore let us represent the integrand in (8) as a Taylor series and restrict ourselves to a few terms. After calculations we obtain

$$1 = \frac{4\pi e^2 \hbar n}{E^2 m} + q^2 \frac{16\pi e^2}{\hbar E^4} \sqrt{\frac{2}{m}} \int_0^\infty \frac{x^{3/2}\,dx}{e^{(x-\mu)/\theta}+1}. \tag{9}$$

Solving this equation with respect to E^2 to accuracy up to q^2 gives an expression determining the energy of the collective oscillations of the system for an arbitrary temperature:

$$E^2 \equiv \hbar^2\omega^2 = \omega_p^2\hbar^2 + \frac{3\hbar^2 k^2}{m E_F^{3/2}} \int_0^\infty \frac{x^{3/2}\,dx}{e^{(x-\mu)/\theta}+1}, \tag{10}$$

where $n = N/\Omega$; $E_F = (3\pi^2)^{2/3}\hbar^2 n^{2/3}/2m$ is the Fermi energy; $q = \hbar k$; $\omega_p = 4\pi e^2 n/m$ is the plasma frequency.

Let us consider two limiting cases. At low temperatures, i.e., temperatures below the degeneracy temperature, we use the expansion of the integrand in powers of θ/E_F. At high temperatures it can be expanded in powers of

$$\frac{n}{2}\left(\frac{2\pi\hbar}{m\theta}\right)^{3/2}.$$

Using the expression given in (12), it is easy to obtain the value of the integral in (10) for the first case:

$$\int_0^\infty \frac{x^{3/2}\,dx}{e^{(x-\mu)/\theta}+1} = \frac{2}{5}E_F^{5/2} + \frac{\pi^2 E_F^{1/2}}{6}\left(\frac{\theta}{E_F}\right)^2. \tag{11}$$

If we introduce the notation $\gamma = E_F/\hbar\omega_p$, then, substituting (11) into (10), we obtain

$$\frac{\hbar\omega}{\hbar\omega_p} = 1 + \frac{6}{5}\gamma^2\left(\frac{k}{k_F}\right)^2 + \frac{\pi^2\gamma^2}{2}\left(\frac{\theta}{E_F}\right)^2\left(\frac{k}{k_F}\right)^2. \tag{12}$$

At zero temperature (12) goes over into the well-known spectrum, which was first obtained by A. A. Vlasov (9), and later by Bohm and Pines (13).

Let us note that, in constructing the model Hamiltonian (4), exchange terms were not taken into account. Their inclusion in (14), without the temperature correction, leads to the spectrum

$$\frac{\hbar\omega}{\hbar\omega_p} = 1 + \frac{6}{5}\gamma^2(1 - 0.208\, r_s)\left(\frac{k}{k_F}\right)^2, \tag{13}$$

where $r_s = r_0/a_0$, $r_0 = (3/4\pi n)^{1/3}$, and a_0 is the Bohr radius. Hence it is seen that the simplification made (neglect of exchange terms) is permissible in the case $r_s < 1$ (high density).

Let us make a remark concerning the cessation of collective oscillations in the system. Physically this is connected with the fact that, at a certain distance from a charge, effective screening of its Coulomb field takes place. This leads to the result that in the system only those collective oscillations can arise whose wavelength is greater than this distance. Since $\lambda = 2\pi/k$, collective oscillations do not arise for sufficiently large k. Mathematically this is expressed in the fact that the isolated root (7) merges with the continuous spectrum. At a temperature different from zero a difficulty arises because of the smearing of the Fermi sphere. However, at sufficiently low temperatures, when the smearing region is small, in the expansion of the Fermi function in a Taylor series we restrict ourselves to two terms. This makes it possible to find the limiting energy.

If we denote $\beta = k'/k_F$, where $\lambda' = 2\pi/k'$ is the minimum wavelength that can propagate in the system, then

$$\beta = 0.47\left(1 - \frac{\theta}{E_F}\right) r_s^{1/2}. \tag{14}$$

At a temperature equal to zero, (14) goes over into the expression obtained by Ferrell [14].

At high temperatures the dimensionless parameter was indicated above. Expansion in this parameter also readily makes it possible to calculate the integral in (10). Finally, retaining two terms in the expansion, we obtain for the frequency of the collective oscillations:

$$\omega^2 = \omega_p^2 + 3\frac{\theta}{m}k^2 - \frac{3\pi^{3/2} n\hbar^3}{4m^{5/2}\theta^{1/2}}\, k^2. \tag{15}$$

The third term on the right-hand side of (15) is of a purely quantum character and vanishes in the classical limit $\hbar \to 0$. The result for this case was first obtained by A. A. Vlasov [9].

Let us establish the connection between (7) and the dispersion relations in the works [9,10] for the classical case. (7) is easily transformed to the form

$$1 = \frac{8\pi e^2\hbar^2}{\Omega q^2}\sum_p\left\{\frac{n_p\,\dfrac{q^2}{2m}}{(E-\mathbf{pq}/m)^2-(q^2/2m)^2}+A(\mathbf{p},\mathbf{q})+B(\mathbf{p},\mathbf{q})\right\}, \qquad (16)$$

where the second and third terms, after the substitution

$$n_p \to \exp(\mu - E(p))/\theta, \qquad \mathbf{q} = \hbar\mathbf{k}, \qquad \mathbf{p} = m\mathbf{v},$$

turn out to be proportional to $\hbar^2$ and $\hbar^3$. Passing in (16) from the sum to an integral and assuming that $\hbar \to 0$, we obtain

$$1 = \frac{4\pi e^2 n}{m}\int d^3v\,\frac{f(v)}{(\omega-\mathbf{kv})^2}, \qquad (17)$$

where $f(v)$ is the Boltzmann distribution.

In this form relation (17) was obtained by Bohm and Gross [10], but it is identical to the relation of A. A. Vlasov [9]. To establish this identity it is sufficient to integrate (15) by parts.

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Note: Figure translations are in progress. See original paper for figures.