Reports of the Academy of Sciences of the USSR

1960

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1960. Volume 135, No. 2

MATHEMATICS

E. Ya. Remez and V. D. Koromyslichenko

VL. MARKOV'S PROBLEM FOR POLYNOMIALS OF A SYSTEM OF CHEBYSHEV FUNCTIONS AND THE NOTION OF A REGULAR T-SYSTEM

(Presented by Academician N. N. Bogolyubov on VI 6, 1960)

1°. Subject of the communication. Formulation of the basic definitions.

In 1892 there appeared a remarkable work by V. A. Markov ([1]), in which there was first considered a problem of considerable theoretical and practical interest: the problem of a rational polynomial of least deviation from zero, in the Chebyshev sense, on a given interval, in which the coefficients of the polynomial are subject to a linear relation of general form. Later Vl. Markov' s problem attracted the attention of a number of researchers; the literature devoted to it continues to grow uninterruptedly down to our time. Among the works in this direction, which have introduced new elements into the general development of the problem, we note here in particular the investigations ([2–4]).

Questions naturally arise concerning the extension of this subject matter to polynomials of more general systems of Chebyshev functions (T-systems, first introduced into consideration by S. N. Bernstein ([5])); these have been investigated to a considerably lesser extent. In the present note, which is a preliminary communication by the authors, there is indicated a natural specialization of the concept of a T-system, corresponding to the essence of the problem, which makes it possible to single out important classes of T-systems for which the methods or results of the theory of rational polynomials of Vl. Markov admit the most far-reaching generalization.

The known definition (([5]), Chap. I) of a T-system of functions $\{\varphi_\nu(x)\}_0^n$, continuous on the interval $[a, b]$, which we shall here regard as closed but not necessarily finite, requires that every polynomial

$$F(x) = \sum A_\nu \varphi_\nu(x) \quad \left(\sum |A_\nu| > 0\right)$$

have no more than n zeros on $[a, b]$, thus ensuring the preservation of one of the essential properties of the rational polynomial

$$p_n(x) = \sum_0^n A_\nu x^\nu.$$

Bearing in mind the preservation of another property of the polynomials $p_n(x)$, especially essential here for us, consisting in the known ([1]) restriction on the possible number for them of ($\leq n+1$) points at which the modulus-maximum

$$L = L[p_n]$$

is attained—the so-called "points of (maximum) deviation (from zero)"—we shall introduce the following specialized basic definitions:

Definition 1. A T-system $\{\varphi_\nu(x)\}_0^n$ $(a \leq x \leq b)$ will be called **regular** if every one of its polynomials $F(x) \not\equiv \text{const}$ has no more than $n+1$ points of deviation on $[a, b]$.* The formulation immediately extends—

* In the cases of an infinite interval $[a, b]$, in order to preserve also the basic facts of connection with the theory of orthogonal polynomials ([3]), we shall, in general, additionally assume the existence of the integrals

$$\int_a^b \varphi_\nu(x)\,dx \quad (\nu = 0, \dots, n)$$

in conjunction with conditions of the type $\varphi_\nu(\pm\infty) = 0$.

also extends to the case of a T-system with fixed zeros at one or both of the points a, b, and also to a "periodic T-system" ([5], p. 49, 11; [7], pp. 69, 68); in the periodic case the maximum admissible number of deviation points on the exact (half-closed) period $[a, b]$ decreases to $n = 2m$.

Definition 2. If a regular T-system, ordinary or with one or two fixed zeros, is a TM-system ([7]) (i.e., each of its consecutive segments $\{\varphi_\nu(x)\}_0^k$ is a T-system), we shall call it a **regular TM-system**. When this definition is extended to the periodic case, only the segments $\{\varphi_\nu(x)\}_0^{2r}$ $(r = 0, \dots, m-1)$ are, naturally, to be taken into account.

To justify these definitions, one must verify, first, that the concept of a regular T-system is already more general than the concept of a T-system, and, second, that the scope of the notions introduced by our definitions is nevertheless sufficiently broad in practice.

2^0. **Existence of irregular T-systems of any prescribed order n, possessing polynomials with an arbitrarily large number of deviation points.** We begin by constructing an example of a two-member TM-system $\{\varphi_\nu(x)\}_0^1$, a polynomial of which may have an arbitrarily large, or even infinite, number of deviation points without becoming identically constant.

Take $[a, b] = [0, 1]$ and, for an arbitrary choice of a natural number N and a proper fraction $\varepsilon < 0$,

$$\varphi_0(x) = 1 + \frac{\varepsilon}{2N\pi} \sin 2N\pi x, \qquad \varphi_1(x) = x. \tag{1}$$

These two functions, as is not hard to verify, form a TM-system of order $n = 1$. This T-system is not regular for $N > 2$, since each of its polynomials obtained for $A_1 = 0$ has N deviation points. The example may be modified by replacing, for $0 \le \nu_1 < \nu_2 \le N - 1$, $\varphi_0(x)$ by

$$\bar{\varphi}_0(x) = \begin{cases} \varphi_0(x), & \text{for } 0 \le x < \dfrac{1}{4N} + \dfrac{\nu_1}{N}, \\ 1 + \dfrac{\varepsilon}{2N\pi}, & \text{for } \dfrac{1}{4N} + \dfrac{\nu_1}{N} \le x \le \dfrac{1}{4N} + \dfrac{\nu_2}{N}, \\ \varphi_0(x), & \text{for } \dfrac{1}{4N} + \dfrac{\nu_2}{N} < x \le 1. \end{cases} \tag{2}$$

Here the corresponding polynomials have, in addition to $\nu_1 + N - \nu_2 - 1$ isolated deviation points, a continuum of deviation points filling the middle part of the interval $[0, 1]$.

Taking into account, further, the possibility of successively extending the constructed TM-system under an arbitrarily small narrowing of the interval $[a, b] = [0, 1]$, which follows from a result of M. A. Rutman (see ([6]), p. 64), we are convinced of the existence of an extended TM-system of any order n, whose polynomials with $A_1 = A_2 = \dots = A_n = 0$ have either $N > n + 1$ deviation points or even a continuum-infinite number of deviation points. If, however, we replace the indicated T-system $\{\varphi_\nu(x)\}_0^n$ by some "equivalent" ([5,7]) T-system $\{\psi_\nu(x)\}_0^n$ (no longer necessarily a TM-system), applying to the former one or another nonsingular linear transformation, we can easily arrange matters so that, in the expressions of the polynomials $\sum A'_\nu \psi_\nu(x)$ with an arbitrarily large number of deviation points, all A'_ν ($\nu = 0, \dots, n$), or any preassigned part of them, are nonzero.

Let us add that, by modifying the specification of the function $\varphi_0(x)$ in (1) somewhat differently, one could obtain T-systems with polynomials for which the set of deviation points is countably infinite.

3°. Two sufficient criteria allowing one to establish the regularity of T-systems in fairly broad classes of cases. The notion of a T-system regular in the strengthened sense.

I. *Regularity of a T-system $\{\varphi_\nu(x)\}_0^n$ $(a \le x \le b)$ is ensured in all cases when among its polynomials $F(x)$ there is some*

$$F^*(x) = \sum A_\nu^*(x)\varphi_\nu(x) \equiv 1.$$

Indeed, let ι, η be, respectively, the number of interior and boundary deviation points of some arbitrarily chosen polynomial $F(x)$ of the given T-system; $\iota = \iota_0 + \iota_1$, $\eta = \eta_0 + \eta_1$, where the lower index $s = 0$ or $s = 1$ refers to the number of deviation points at which $F(x) = (-1)^s L$. The polynomial $F(x) - (-1)^s LF^*(x)$ has, evidently, on the interval $[a, b]$ ι_s "double" ((5), p. 8) roots and η_s "simple" roots $(s = 0, 1)$. Hence we derive ((5), p. 8, Lemma 1) the inequalities $2\iota_0 + \eta_0 \le n$, $2\iota_1 + \eta_1 \le n$; adding them, we obtain $2\iota + \eta \le 2n$, $\iota + \eta \le \frac{1}{2}(2\iota + \eta) \le n + 1$.

A slight modification of the argument makes it possible to extend criterion I also to the case of a periodic T-system. By virtue of the criterion established here, the regularity of most of the important T-systems cited in (5-7) as examples is immediately revealed (as a rule, they are also TM-systems), since in them $F^*(x) \equiv 1$ occurs among the basic functions $\varphi_\nu(x)$ $(\nu = 0, \dots, n)$. Equally regular is, for example, the TM-system $\{x^\nu / q(x)\}_0^n$ $(a \le x \le b)$, where $q(x) = \sum_0^n k_i x^i > 0$ on $[a, b]$: here $F^*(x) = \sum k_\nu \varphi_\nu(x) = q(x)/q(x) \equiv 1$. But criterion I is obviously inapplicable to cases of T-systems with fixed zeros.

II. *For a T-system $\{\varphi_\nu(x)\}_0^n$ $(a \le x \le b)$ with possible ξ fixed zeros* ($\xi = 1, 2$ *or* 0)*, regularity is ensured also in the case when the polynomials $F(x)$* ($\not\equiv$ const) *possess, in the open interval (a, b), a derivative $F'(x)$ which vanishes at no more than $n - 1 + \xi$ points.*

Indeed, under the given condition we certainly have $\iota \le n - 1 + \xi$, $\eta \le 2 - \xi$, $\iota + \eta \le n + 1$.

Examples of regular (by criterion II) TM-systems with fixed zeros.

1) $\{x^{\alpha_\nu}\}_{\nu=0}^n$ $(0 < \alpha_0 < \alpha_1 < \dots < \alpha_n)$ on $[0, k]$ $(0 < k < \infty)$.

2) $\{x^{\alpha_0}, \dots, x^{\alpha_{n-1}}, x^{\alpha_n} f(x)\}$ (5) $(0 < \alpha_0 < \dots < \alpha_n)$ on $[0, k]$, if $f(x), f'(x), \dots, f^{(n)}(x)$ are nonnegative on $(0, k)$ and nowhere vanish simultaneously.

3) $\{\mathrm{sh}(\nu + 1)x\}_{\nu=0}^n$ on any $[0, k]$.

4-6) $\{\sin(\nu + 1)x\}$, $\{\sin(\nu + \frac{1}{2})x\}$, $\{\cos(\nu + \frac{1}{2})x\}$—each for $\nu = 0, \dots, n$, on $[0, \pi]$.

7) $\{e^{-\frac{1}{2}x^2} x^\nu\}_{\nu=0}^n$ on $[-\infty, +\infty]$ (cf. (8)).

8) $\{e^{-\frac{1}{2}x} x^\nu\}_{\nu=0}^n$ on $[0, \infty]$.

9) $\{(1 - x)^{\alpha/2}(1 + x)^{\beta/2} x^\nu\}_{\nu=0}^n$ $(\alpha, \beta > 0)$ on $[-1, 1]$.

10) $\{e^{s_0 x}, x e^{s_0 x}, \dots, x^{m_0 - 1} e^{s_0 x},\ e^{s_1 x}, \dots, x^{m_1 - 1} e^{s_1 x}, \dots, e^{s_r x}, \dots, x^{m_r - 1} e^{s_r x}\}$,

where $\sum m_j = n + 1$; $0 > s_0 > s_1 > \dots > s_r$ on $[a, \infty]$ $(-\infty < a < +\infty)$.

Definition 3. A T-system, ordinary or generalized ((5); (7), pp. 69 and 68), regular on $[a, b]$, will be called *regular in the strengthened sense* if it satisfies, with respect to the existence of the derivative $F'(x)$ and the number of its zeros, either the conditions of criterion II, or, in the periodic case, analogous conditions with replacement of $n-1+\xi$ by $n\ (=2m)$ and of (a, b) by $[a, b]$.

Let us denote by $\kappa = \iota + \eta$ the number of deviation points x_s of an arbitrarily chosen nondegenerate polynomial $F(x)$ of the system $\{\varphi_\nu(x)\}_0^n$ on $[a, b]$; by

ϑ is the number of sign changes in the sequence $\{F(x_s)\}_1^\chi$ $(x_1 < x_2 < ... < x_\chi)$, and $r (= (\chi - 1) - \vartheta)$ is the number of repetitions of sign. In the case of a (nonperiodic) T-system, regular in the strengthened sense, the remarkable relation (cf. (3), in the case $[x^\nu]_0^n$ with $\zeta = 0$) turns out to be applicable:

$$(\chi - \eta) + r \leqslant n - 1 + \zeta \tag{3}$$

(respectively $(\chi - \frac{1}{2}\eta) + r \leqslant n$ in the periodic case). In particular, for $\chi = n + 1$ this implies the existence of an alternance $(r = 0)$.

A straightforward verification shows that the conditions of Definition 3 are satisfied not only by the T-systems of examples 1)-10) (which was already provided for by the very formulation of Definition 3), but also by most of the most important T-systems without fixed zeros mentioned above (cf. ([5]-[7])).

The system $\varphi_0(x) = \dfrac{1}{1+x}$, $\varphi_1(x) = x$ $(0 \leqslant x \leqslant 1)$ may serve as a simple example of a regular T-system for which relation (3) is violated (for $A_0 = 1,\ A_1 = -1/2$) and which, consequently, is certainly not regular in the strengthened sense.

4°. Extension to regular T-systems of one of the fundamental facts pertaining to the classical problem of V. A. Markov. In order to explain here more closely, on one fundamental fact, the role of the principal concept of regularity defined by us in § 1°, we give the formulation of a proposition established by a direct generalization of the argument used by E. Ya. Remez (([9]), pp. 313-319) in the case $\{x^\nu\}_0^n$:

If the generalization of V. A. Markov' s problem for a regular T-system $\{\varphi_\nu(x)\}_0^n$ is rewritten in terms of the general problem of Chebyshev approximation (([7]), p. 108) for a (continuously infinite) system of inconsistent equations with n free parameters-unknowns, then the inconsistent system of equations just mentioned will always have a unique Chebyshev subsystem (certainly irreducible), provided the existence of at least one nondegenerate Chebyshev solution is assumed ($F^{(n)}(x) \neq$ const).

Institute of Mathematics
Academy of Sciences of the Ukrainian SSR

Received
19 V 1960

REFERENCES CITED

[1] Vl. Markov, *On functions least deviating from zero*, St. Petersburg, 1892. [2] A. Psheborkii, Reports of the Kharkov Mathematical Society, **14**, 65 (1913). [3] Ya. A. Shokhat, *Investigation of one class of polynomials*, Ekaterinburg, 1918. [4] E. V. Voronovskaya, Proceedings of the Leningrad Industrial Institute, No. 4, 22 (1937); Proceedings of the Third All-Union Mathematical Congress, 3, Moscow, 1958. [5] S. N. Bernstein, *Extremal Properties of Polynomials*, 1937. [6] M. G. Krein, UMN, **6**, No. 4 (1951). [7] E. Ya. Remez, *General computational methods of Chebyshev approximation*, Publishing House of the Academy of Sciences of the Ukrainian SSR, 1957. [8] D. G. Grebenyuk, Proceedings of the Institute of Mathematics and Mechanics of the Academy of Sciences of the Uzbek SSR, issue 18 (1956). [9] E. Ya. Remez, Ukrainian Mathematical Journal, **9**, No. 3 (1957).

Note: Figure translations are in progress. See original paper for figures.