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Abstract

Full Text

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ON REGULARIZATION AND THE DIVISION PROBLEM

(Presented by Academician P. S. Aleksandrov on 15 I 1960)

In the present note we set forth results relating to the division problem and the problem of canonical regularization. We shall use the spaces of type S introduced in (2), as well as the notation K and S adopted in (2) for the spaces D and G introduced by Schwartz (3).

The division problem, formulated by Schwartz (3), consists in the question of the solvability of the equation

$$FU = T, \quad T \in K', \tag{1}$$

in the space K' , where F is a given infinitely differentiable function. This problem was solved by Schwartz in the case of one independent variable, and in the general case was studied by a number of authors (4-8). The greatest progress was achieved by Hörmander (6), who showed that in the case where the function F is a polynomial, equation (1) has a slowly increasing generalized solution (i.e., a functional on S) if the right-hand side is a slowly increasing generalized function.

As we shall show below, over the space K division by any function analytic in a neighborhood of a real manifold is possible. It is not hard to show that this result, generally speaking, is false for the space S .

The reasoning is based on a theorem connecting the division problem with the regularization problem.

Theorem 1. *Equation (1) is solvable in K' if and only if, for every n , there exists a regularization of the function $\frac{1}{(F\bar{F})^n}$ satisfying the condition*

$$(F\bar{F})^n \left[\frac{1}{(F\bar{F})^n} \right] = 1. \tag{2}$$

A special construction makes it possible to obtain the following result.

Theorem 2. *Let p be a function of the form*

$$p = p(y, x_1, \dots, x_n) = y^m + p_1(x_1, \dots, x_n)y^{m-1} + \dots + p_m(x_1, \dots, x_n). \tag{3}$$

There exists a regularization of the function $\frac{1}{p}$ satisfying condition (2), if there exists, satisfying the same condition, a regularization of the function $\frac{1}{D_p}$, where, if

$$p(y, x) = \prod_{j=1}^m [y - y_j(x)],$$

then

$$D_p(x) = \prod [y_i(x) - y_j(x)] \quad (x = (x_1, \dots, x_n)),$$

where for each x the product is taken over all pairs of roots that do not coincide identically in any neighborhood of the point x .

A function $A = A(y, x)$, analytic in a neighborhood of a real manifold, by the Weierstrass theorem ⁽⁹⁾ in every bounded domain—

of G is represented in the form

$$A = Wp,$$

where W is a function not vanishing in G ; p is a polynomial of the form (3) with analytic coefficients. In the domain G put

$$\left[\frac{1}{A} \right] = \frac{1}{W} \left[\frac{1}{p} \right].$$

Thus, by means of Theorem 2, the question of regularization of the function A^{-1} of $n + 1$ variables is reduced to the question of regularization of the function D_p^{-1} of n variables. Moreover, since A is analytic in a neighborhood of the real manifold, D_p is also analytic in a neighborhood of the real manifold. Carrying out induction on the number of dimensions, we construct a regularization of the function A^{-1} satisfying condition (2).

Theorem 3. *Over the space K , division by any function analytic in a neighborhood of the real manifold is possible.*

From the proof of this theorem one can also obtain a result of L. Hörmander.

Theorem 3 is valid not only for the space $K = S_0$, but also for any space S_0^α , $\alpha > 1$.

The question of division by a function of the form (3) becomes considerably more complicated if the coefficients $p_1(x), \dots, p_m(x)$ are arbitrary infinitely differentiable functions. The following results have been established:

Theorem 4. Let p be a function of the form (3) with infinitely differentiable coefficients, vanishing only at the origin of the coordinates. If among the roots of this polynomial there is a root y_1^+ (y_1^-), $\text{Im } y^+(x) \geq 0$ ($\text{Im } y^-(x) \leq 0$), such that the inequality

$$\prod_j |y_1^+(x) - y_j^-(x)| \leq C_q |x|^q \quad \left(\prod_j |y_1^-(x) - y_j^+(x)| \leq C_q |x|^q \right),$$

where the product is taken over all roots $y_j^-(x)$ ($y_j^+(x)$) for which $\text{Im } y_j^-(x) \leq 0$ ($\text{Im } y_j^+(x) \leq 0$), is fulfilled for all $q > 0$ and for some $C_q > 0$, then the function $\frac{1}{p}$ is not regularizable over the space K .

From Theorem 1 follows the impossibility of division by such a polynomial p .

Theorem 5. If the roots of the polynomial p of the form (3) satisfy the condition: either $\text{Im } y_i(x) \equiv 0$, or $|\text{Im } y_i(x)| > 0$, then division by the polynomial p is possible.

Theorem 6. Independently of the behavior of the roots of the polynomial p , division by this polynomial is possible over the space S^β , if the index β_1 , corresponding to the variable y , is less than or equal to 1.

We apply the results obtained to certain classes of differential and integral equations.

Theorem 7. Let $A(s)$ be a matrix whose elements are functions analytic in a neighborhood of the real manifold. Then, if T is an arbitrary vector-functional belonging to the spaces $*Z'$, E' , $S_a^{\alpha'}$, $\alpha > 1$, then the system

$$A \left(i \frac{\partial}{\partial x} \right) U = T \tag{4}$$

or the equivalent system

$$\tilde{A}(s) * U = T$$

* E is the space introduced by Schwartz of all infinitely differentiable functions with the topology in which co

have solutions in vector functionals respectively over the spaces Z' , E' , $S_\beta^{\alpha'}$.

A particular case of system (4) is a differential-difference system in partial derivatives, and a particular case of system (5) is a system of integral equations with a kernel depending on the difference of the arguments and decreasing no more slowly than an exponential of first order.

Theorem 8. The equation

$$\left[\frac{\partial^m}{\partial y^m} + a_1(x) \frac{\partial^{m-1}}{\partial y^{m-1}} + \dots + a_m(x) \right] U = T$$

with right-hand side belonging to the space S'_β , has a solution belonging to the same space if the index β_1 is less than or equal to 1.

In other words, one ensures only the existence of a solution growing no faster than an exponential of first order. A slowly growing solution, i.e., a solution that is a functional over $Z = S^0$, may also fail to exist, as Theorem 4 shows. Let us note that it can be shown that, generally speaking, there is also no solution growing no faster than an exponential of order less than 1.

We now turn to the problem of canonical regularization. By the definition given by I. M. Gelfand and G. E. Shilov ⁽¹⁾, a regularization of a function $F = F(x)$ is called **canonical** if the following relations hold:

- I. $h[F] = [hF]$, $h \in C^\infty$.
- II. $[F_1] + [F_2] = [F_1 + F_2]$.
- III. $\frac{\partial}{\partial x_i}[F] = \left[\frac{\partial}{\partial x_i} F \right]$, $i = 1, \dots, n$.

I. M. Gelfand and G. E. Shilov solved the problem of canonical regularization in the case of one independent variable. The general case proved to be considerably more difficult.

We formulate a result obtained by us earlier.

Theorem 9. There exists a canonical regularization of functions of the form $\frac{h}{p}$, where h is an arbitrary function belonging to C^∞ , and p is an arbitrary polynomial of the form (3) such that, for some distribution of the indices + and – among the roots whose imaginary part is identically zero, the function

$$D_p^*(x) = \prod_{i,j} [y_i^+(x) - y_j^-(x)]$$

does not vanish anywhere.

When p is an arbitrary polynomial of the form (3), canonical regularization of functions of the form $\frac{h}{p}$ does not exist, as V. V. Grushin showed on the example of the function $\frac{1}{r^2}$. Moreover, the following is true.

Theorem 10. For any h there exist infinitely many algebraically independent polynomials p in n variables such that the function $\frac{1}{p}$ is not canonically regularized.

However, the construction built in the proof of Theorem 2 makes it possible to establish the following fact:

Theorem 11. Functions of the form $\frac{h}{A}$, where $h \in C^\infty$, and A is an analytic function in a neighborhood of a real manifold, are regularized “almost canonically” in the sense that condition III is satisfied only with respect to one variable.

It is possible to establish a connection between the canonical regularizability of the function $\frac{1}{A}$ in a certain strong sense and divisibility by the function A in the generalized sense, analogous to the one that we established between regularizability with fulfillment of condition (2) and ordinary divisibility. Namely, the following theorem holds:

Theorem 12. *The following three conditions are equivalent:*

1. *For any regularization of the function $\frac{1}{C}$ over the perfect space Φ and a fixed regularization of the function $\frac{1}{B}$, there exists a regularization of the function $\frac{1}{A}$, $A = BC$, such that*

$$C \left[\frac{1}{A} \right] = \left[\frac{1}{B} \right], \quad B \left[\frac{1}{A} \right] = \left[\frac{1}{C} \right]$$

(canonicity in the strong sense).

2. *The mapping $B\varphi \rightarrow \varphi$ is continuous in the topology Φ_C , where Φ_C is the quotient space $\Phi/C\Phi$ with the corresponding topology, and the subspace $C\Phi$ is assumed to be closed.*
3. *The equation*

$$BU = T$$

is solvable in Φ'_C for every right-hand side T belonging to Φ'_C .

It has been possible to show that one of these conditions, and hence all the others, is fulfilled only in the following two special cases ($\Phi = K$):

1. The functions B and C have only isolated zeros of finite order.
2. The functions B and C satisfy the conditions of Theorem 5.

In all likelihood, these conditions are fulfilled when B and C are arbitrary analytic functions in a neighborhood of a real manifold.

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