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Abstract

Full Text

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ON THE CAUCHY PROBLEM IN THE LARGE FOR CERTAIN NONLINEAR DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

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In the present note the Cauchy problem is considered for the equation

$$u_t + \varphi(u_x) = 0, \qquad \varphi''(v) \geq a > 0, \qquad \varphi(0) = \varphi'(0) = 0 \tag{1}$$

in the half-plane $t \geq 0$ with the initial condition

$$u(0, x) = u_0(x), \tag{2}$$

where $u_0(x)$ is an arbitrary bounded function.

We shall define a generalized solution of problem (1), (2), prove its existence, uniqueness, continuous dependence on the initial conditions, and clarify certain properties of generalized solutions. One of the characteristic properties of these solutions, substantially distinguishing them from generalized solutions of the Cauchy problem for quasilinear equations of the first order ([1], [2]), is that they are continuous in the half-plane $t > 0$ for arbitrary initial conditions.

As a consequence of the study of the Cauchy problem (1), (2) with discontinuous initial functions, the Cauchy problem for the equation

$$v_t + (\varphi(v))_x = 0, \qquad \varphi''(v) \geq a > 0, \qquad \varphi'(0) = 0 \tag{3}$$

will be considered in the case when the initial conditions may be identified with certain functionals.

Let $\mathcal{L} = \bigcup_k \mathcal{L}(K)$, where $\mathcal{L}(K)$ is the set of bounded functions satisfying the Lipschitz condition on the whole line with constants $\leq K$.

1. The case of a continuous initial function $u_0(x) \in \mathcal{L}$.

Definition 1. A continuous bounded function $u(t, x)$ is called a **generalized solution of the Cauchy problem** (1), (2), if: 1) $u(t, x)$ satisfies the Lipschitz condition in both variables with constants bounded for $t \geq 0$; 2) $u(t, x)$ satisfies equation (1) almost everywhere; 3) $u(0, x) = u_0(x)$; 4) $\Delta u_x / \Delta x \leq C/t$.

Theorem 1. The generalized solution $u(t, x)$ of the Cauchy problem (1), (2) exists and is unique, and

$$u(t,x) = \min_{-\infty<\sigma<\infty} I(\sigma,t,x), \qquad I(\sigma,t,x) = u_0(x - t\varphi'(\sigma)) + t\int_0^\sigma \xi\varphi''(\xi)\,d\xi. \tag{4}$$

The proof of uniqueness is not difficult to obtain by using the theorem on uniqueness of generalized solutions of the Cauchy problem for quasilinear equations (^{1}), observing that $v(t,x) = \partial u/\partial x$ is a generalized solution of the Cauchy problem for equation (3) with initial condition $v_0(x) = u_0'(x)$.

For the proof of existence, consider the function equal to

$$\min_{-\infty<\sigma<\infty} I(\sigma,t,x).$$

Let $\sigma(t, x)$ be the leftmost point of minimum of the function $I(\sigma, t, x)$ for fixed t and x. One can show that

$$\sigma(t,x) = \Phi((x - s_+(t,x))/t),$$

where $\Phi(v)$ is the inverse function to $\varphi'(v)$, and $s_+(t, x)$ is the function considered in (^{2}), p. 436. Using certain

properties of the function $S_+(t, x)$ indicated in (2), and the fact that, for $u_0(x) \in \mathcal{L}(K)$, $|\sigma(t, x)| \leq K$, we obtain that the function

$$\min_{-\infty<\sigma<\infty} I(\sigma,t,x)$$

is a generalized solution of the Cauchy problem (1), (2).

Let us consider the connection between the generalized solutions of problem (1), (2) and the solutions of the Cauchy problem for nonlinear parabolic equations

$$u_t + \varphi(u_x) = \varepsilon u_{xx}, \qquad \varphi(0) = 0, \qquad \varepsilon > 0. \tag{5}$$

Theorem 2. *There exists a unique bounded function $u_\varepsilon(t, x)$, having for $t > 0$ continuous derivatives entering equation* (5)*, with $|\partial u_\varepsilon/\partial x| \leq K$, satisfying equation* (5) *and the initial condition*

$$u_\varepsilon(0,x) = u_0(x) \in \mathcal{L}. \tag{6}$$

The uniqueness of the function $u(t, x)$ follows from the fact that the difference of two such functions satisfies an equation with the maximum principle.

It is not difficult to show the existence of a solution of the Cauchy problem $(5), (6)$ in the case of a finite initial function $u_0(x) \in C^{(2)}$. Relying on this fact and using Bernstein estimates for the derivatives of solutions of equation (5), one can carry out the construction of a solution of problem $(5), (6)$ also for $u_0(x) \in \mathcal{L}$.

The following assertion is a theorem on the continuous dependence of the solutions of the Cauchy problem $(5), (6)$ on the initial conditions (see (3)).

Theorem 3. *Let $u_0(x)$ and $\tilde{u}_0(x) \in \mathcal{L}(K)$, $|u_0(x)| \leq M$, $|\tilde{u}_0(x)| \leq M$; let $u_\varepsilon(t, x)$ and $\tilde{u}_\varepsilon(t, x)$ be the corresponding solutions of problem $(5), (6)$, with $\varepsilon \leq \varepsilon_0$. Then for any rectangle $R\{|x| \leq l,\ 0 \leq t \leq T\}$ and $\gamma > 0$ one can specify such N and $\delta > 0$, independent of ε, that $|u_\varepsilon(t, x) - \tilde{u}_\varepsilon(t, x)| \leq \gamma$ in R, if $|u_0(x) - \tilde{u}_0(x)| \leq \delta$ for $|x| \leq N$.*

Let us note that in Theorems 2 and 3 the convexity of the function $\varphi(v)$ was not used. If $\varphi(v)$ is convex, then the following holds.

Lemma 1. *Let $u_\varepsilon(t, x)$ be a solution of problem $(5), (6)$, with $u_0(x) \in \mathcal{L}(K)$ and $\varphi''(v) \geq a > 0$ for $|v| \leq K$. Then $\partial^2 u_\varepsilon / \partial x^2 \leq 1/at$.*

Theorem 4. *As $\varepsilon \to 0$, the solutions $u_\varepsilon(t, x)$ of the Cauchy problem $(5), (6)$ converge to the generalized solution of the Cauchy problem $(1), (2)$ uniformly in every rectangle of the half-plane $t \geq 0$.*

It can be shown that the family $\{u_\varepsilon(t, x)\}$, for $\varepsilon \leq \varepsilon_0$, is compact in the space C on any rectangle in the half-plane $t \geq 0$, and, if $u_0(x) \in \mathcal{L}(K)$, the estimates $|\partial u_\varepsilon / \partial x| \leq K$ for $t > 0$ and $|\varepsilon\, \partial^2 u_\varepsilon / \partial x^2| \leq K_1(t_0)$, $|\partial u_\varepsilon / \partial t| \leq K_2(t_0)$ for $t \geq \varepsilon t_0$ hold. In consequence of Lemma 1 and the estimate $|\partial u_\varepsilon / \partial x| \leq K$, we obtain that

$$\varepsilon \int_a^b \left| \frac{\partial^2 u_\varepsilon}{\partial x^2} \right| dx \to 0$$

as $\varepsilon \to 0$ uniformly for $t \geq \delta > 0$. On the basis of the results of $(1, 3)$, $\{\partial u_\varepsilon / \partial x\}$ converges as $\varepsilon \to 0$ in the norm L_1 on each rectangle of the half-plane $t > 0$ to a bounded function $v(t, x)$, with $\Delta v / \Delta x \leq C/t$.

Relying on these properties of the sequence $\{u_\varepsilon(t, x)\}$ and using the uniqueness of the generalized solution of problem $(1), (2)$, one can prove the assertion of Theorem 4.

Theorems on the stability of generalized solutions of problem $(1), (2)$ with respect to changes of the initial functions will be given below in a more general case of initial conditions.

2. The case of an arbitrary bounded initial function $u_0(x)$.

Definition 2. A bounded function $u(t, x)$ is called a **generalized solution** of problem $(1), (2)$ if: 1) in every half-plane $t \geq \delta > 0$ the function $u(t, x)$ satisfies a

Lipschitz condition in both variables with bounded constants; 2) $u(t,x)$ satisfies equation (1) almost everywhere; 3)

$$\lim_{t\to 0} u(t,x) = \mu(x) = \lim_{\gamma\to 0} \inf_{\xi\in[x-\gamma,x+\gamma]} u_0(\xi)$$

($\mu(x)$ is the lower Baire function for $u_0(x)$); 4) $\Delta u_x/\Delta x \le C/t$.

Theorem 5. The function

$$u(t,x) = \min_{-\infty<\sigma<\infty} I(\sigma,t,x), \qquad I(\sigma,t,x) = \mu(x - t\varphi'(\sigma)) + t\int_0^\sigma \xi\varphi''(\xi)\,d\xi \quad (7)$$

is a generalized solution of the Cauchy problem (1), (2) with initial function $u_0(x)$.

From the definition of the function $u(t,x)$ there follows the following assertion, with the aid of which it is not difficult to show that $u(t,x)$ satisfies requirements 1) and 3) of a generalized solution.

Lemma 2. Let $|u_0(x)| \le M$; let $K_+(t)$ and $K_-(t)$ be the roots of the equation

$$\int_0^K \xi\varphi''(\xi)\,d\xi = \frac{2M}{t}.$$

Then

$$u(t,x) = \min_{K_-(t)\le\sigma\le K_+(t)} I(\sigma,t,x)$$

and

$$\lim_{t\to 0} t\varphi'(K_-(t)) = \lim_{t\to 0} t\varphi'(K_+(t)) = 0.$$

Applying the following Lemma 3 and using Theorem 1, we obtain that requirements 2) and 4) of a generalized solution are fulfilled for $u(t,x)$.

Lemma 3. Let

$$u(t,x) = \min_{-\infty<\sigma<\infty} I(\sigma,t,x), \qquad u^\alpha(t,x) =$$

$$= \min_{-\infty<\sigma<\infty} \left[u(\alpha, x - t\varphi'(\sigma)) + t\int_0^\sigma \xi\varphi''(\xi)\,d\xi \right].$$

Then

$$u^{\alpha}(t,x) = u(t+\alpha, x).$$

For the proof, consider the function

$$G(\sigma, s) = \mu(x - t\sigma - \alpha s) + t\int_0^{\Phi(\sigma)} \xi\varphi''(\xi)\,d\xi + \alpha\int_0^{\Phi(s)} \xi\varphi''(\xi)\,d\xi$$

for fixed x and $t > 0$. It is not hard to see that the assertion of the lemma is equivalent to the equality

$$\min_{-\infty<\sigma,s<\infty} G(\sigma, s) = \min_{-\infty<\sigma<\infty} G(\sigma,\sigma).$$

Using the convexity of the function

$$g(\sigma) = \int_0^{\Phi(\sigma)} \xi\varphi''(\xi)\,d\xi,$$

one can show that

$$G(\sigma, s) \geq G(\sigma_0, \sigma_0),$$

where

$$\sigma_0 = \frac{t\sigma + \alpha s}{t+\alpha}.$$

Hence

$$\min_{-\infty<\sigma,s<\infty} G(\sigma, s) \geq \min_{-\infty<\sigma<\infty} G(\sigma,\sigma);$$

the inequality in the other direction is obvious.

Theorem 6. Let $\{u_n(x)\}$ be a nondecreasing sequence of discontinuous functions, with $|u_n(x)| \leq M$, $u_n(x) \in \mathcal{L}(K_n)$; let $u_n(t,x)$ be the corresponding generalized solutions of the Cauchy problem (1), (2) according to Definition 1. Then $\{u_n(t,x)\}$ converges uniformly on every rectangle in the half-plane $t > 0$ to the function

$$u(t,x) = \min_{-\infty<\sigma<\infty}\left[\mu(x - t\varphi'(\sigma)) + t\int_0^{\sigma} \xi\varphi''(\xi)\,d\xi\right],$$

where

$$\mu(x) = \lim_{n\to\infty} u_n(x).$$

Using representation (4) for the functions $u_n(t,x)$, one can obtain pointwise convergence of the nondecreasing sequence $\{u_n(t,x)\}$ to $u(t,x)$. But since $u(t,x)$ is continuous for $t>0$, the sequence $\{u_n(t,x)\}$ converges to $u(t,x)$ uniformly in every rectangle of the half-plane $t>0$.

Theorem 7. The generalized solution of the Cauchy problem (1), (2) with an arbitrary bounded initial function $u_0(x)$ is unique.

Let $u(t,x)$ be some solution for the initial function $u_0(x)$. From the definition of a generalized solution, $u^\alpha(x) = u(\alpha,x) \in \mathcal{L}(K_\alpha)$ for every $\alpha>0$. Since $\varphi(v)\geq 0$, it follows from equation (1) that $\{u^\alpha(x)\}$ is a nondecreasing sequence as $\alpha\to 0$. Denote by $u^\alpha(t,x)$ the generalized solution of the Cauchy problem (1), (2) with initial condition $u^\alpha(x)$ according to Definition 1; by Theorem 1,

$$u^\alpha(t,x) = u(t+\alpha,x).$$

Applying Theorem 6 to the sequence $\{u^\alpha(t,x)\}$, and taking into account that

$$\lim_{\alpha\to 0} u^\alpha(x) = \mu(x),$$

we have

$$u(t,x) = \lim_{\alpha\to 0} u(t+\alpha,x) = \lim_{\alpha\to 0} u^\alpha(t,x) =$$

$$= \min_{-\infty<\sigma<\infty}\left[\mu(x - t\varphi'(\sigma)) + t\int_0^\sigma \xi\varphi''(\xi)\,d\xi\right].$$

Thus, it has been shown that initial functions having the same lower Baire function correspond to one and the same generalized solution of the Cauchy problem (1), (2).

Theorem 8. Let $\mu_1(x)$ and $\mu_2(x)$ be lower Baire functions for $u_1(x)$ and $u_2(x)$, $|\mu_1(x)|\leq M$, $|\mu_2(x)|\leq M$; let $u_1(t,x)$, $u_2(t,x)$ be generalized solutions of the Cauchy problem (1), (2) with initial functions $u_1(x)$, $u_2(x)$, respectively. Then, for any rectangle $R\{|x|\leq l,\ 0\leq t\leq T\}$ and $\gamma>0$, one can indicate an N such that

$$|u_1(t,x) - u_2(t,x)| \leq \gamma$$

in R, if

$$|\mu_1(x) - \mu_2(x)| \leq \gamma$$

for $|x|\leq N$.

The proof follows from (7), if Lemma 2 is used.

Theorem 8 shows that the solutions of the Cauchy problem (1), (2) have a finite domain of dependence on the initial conditions. Theorem 6 may also be regarded as a theorem on the stability of generalized solutions of problem (1), (2) with respect to changes in the initial functions.

From (7) it is not difficult to obtain the following property of generalized solutions:

Theorem 9. Let $u(t, x)$ be a generalized solution of problem (1), (2) with initial condition $u_0(x)$, $|u_0(x)| \leq M$,

$$\mu_0 = \inf_{-\infty<x<\infty} u_0(x).$$

Then $u(t, x) \to \mu_0$ as $t \to \infty$, uniformly on every interval $|x| \leq l$.

3. On the Cauchy problem for the quasilinear equation (3).

On any $[a, b]$ consider the space $A_{[a,b]}$ of absolutely continuous functions with norm

$$\|\psi\| = |\psi(a)| + \int_a^b |\psi'(x)|\, dx.$$

With the aid of a bounded lower semicontinuous function $\mu(x)$, given on the whole line, define a linear continuous functional on each of the spaces $A_{[a,b]}$ by the formula

$$(f, \psi) = -\int_a^b \mu(x)\psi'(x)\, dx + \mu(b)\psi(b) - \mu(a)\psi(a).$$

We note that in those cases when $\mu(x)$ is continuous or has bounded variation on $[a, b]$, the functional f is a functional of measure type, i.e.

$$(f, \psi) = \int_a^b \psi(x)\, d\mu(x).$$

Definition 3. A generalized solution of the Cauchy problem for equation (3) with initial condition f is a measurable function $v(t, x)$, bounded for $t \geq \delta > 0$, such that: 1) for any piecewise-smooth contour Γ in the half-plane $t \geq 0$,

$$\oint_\Gamma v\, dx - \varphi(v)\, dt = 0,$$

and moreover

$$\int_a^b v(0, x)\, dx = \mu(b) - \mu(a) \quad (^{3,4});$$

2)

$$\Delta v / \Delta x \leq C/t.$$

The corresponding theorems on the existence and uniqueness of such generalized solutions, and on their stability with respect to changes in the initial data, can be obtained from the preceding results, relying on the fact that

$$v(t,x) = \partial u/\partial x,$$

where $u(t,x)$ is a generalized solution of the Cauchy problem (1), (2) with initial condition $u_0(x) = \mu(x)$.

It is not difficult to show that

$$\int_a^b v(t,x)\psi(x)\,dx \to (f,\psi)$$

as $t \to 0$ for $\psi \in A_{[a,b]}$, for arbitrary a and b.

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Note: Figure translations are in progress. See original paper for figures.