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Abstract

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DEFINING RELATIONS OF FINITE SEMIGROUPS OF PARTIAL TRANSFORMATIONS

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1°. In the present note a system of defining relations is found for the semigroup of all partial transformations of a finite set containing $n \geq 4$ elements, and also for its subsemigroup of all one-to-one partial transformations.

Terms: a generating set of a semigroup, an irreducible generating set, a relation of a semigroup with respect to a generating set, a consequence of relations, and a system of defining relations of a semigroup are used in the usual sense (see, for example, ([2])). If a relation $u = v$ of a semigroup is a consequence of the relations Σ of this semigroup, then we shall say that u is **reduced to** v **by means of** Σ.

2°. Let Ω be the set of the numbers $1, 2, \dots, n$, $n \geq 4$; let Δ_1 and Δ_2 be subsets of Ω, where Δ_1 and Δ_2 may be empty. A mapping of Δ_1 into Δ_2 is called a **partial transformation** of the set Ω. If, under the partial transformation a, the number i is mapped to k, we shall write $ai = k$. The totality W_n of all partial transformations of the set Ω and the totality V_n of all one-to-one partial transformations of the set Ω, with respect to the usual multiplication of partial transformations, are semigroups. Denote by H_n the semigroup of all transformations of the set Ω, and by S_n the group of all one-to-one transformations in H_n. Introduce the following notation:

$$a_1 = \begin{pmatrix} 2 & 3 & \cdots & n \\ 2 & 3 & \cdots & n \end{pmatrix}, \qquad a = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 1 & 3 & \cdots & n \end{pmatrix},$$

$$c_i = \begin{pmatrix} 1 & 2 & \cdots & i-1 & i & i+1 & \cdots & n \\ i & 2 & \cdots & i-1 & 1 & i+1 & \cdots & n \end{pmatrix} \qquad (2 \leq i \leq n).$$

It is known that the set M_1 of all $c_2, c_3, \dots, c_n$ is an irreducible generating set of the group S_n, and the set M_2 of all $c_2, c_3, \dots, c_n, a$ is an irreducible generating set of the semigroup H_n ([1]). Let M_3 be the set of all $c_2, c_3, \dots, c_n, a_1$, and M_4 the set of all $c_2, c_3, \dots, c_n, a, a_1$. It is not hard to show that M_3 and M_4 are irreducible generating sets, respectively, of the semigroups V_n and W_n.

3°. Let $c_2^2 = e,\ c_i a_1 c_i = a_i$. Consider the following system of relations of the semigroup V_n with respect to the set M_3:

1. Defining relations of the group S_n with respect to M_1 (see (^{2})).
2. $a_1 e = e a_1 = a_1, \quad a_1 a_2 = a_2 a_1, \quad a_1^2 = a_1$.
3. $a_2 c_i = c_i a_2, \quad a_i c_2 = c_2 a_i \quad (3 \le i \le n)$.
4. $c_2 a_1 a_2 = a_1 a_2$.

$$(\Sigma_1)$$

4°. The following three lemmas can be proved.

Lemma 1. If $ui = k$ $(u \in S_n,\ i \in \Omega)$, then the relation

$$u a_i = a_{ku}$$

of the semigroup V_n is a consequence of the relations (Σ_1).

Lemma 2. The relations

$$a_i a_k = a_k a_i, \qquad a_i^2 = a_i \qquad (1 \le i, k \le n)$$

of the semigroup V_n are consequences of the relations (Σ_1).

Lemma 3. Let $i_1, i_2, \dots, i_n$ be a permutation of the numbers $1, 2, \dots, n$, and let $2 \le m \le n$. If, for u, v from S_n, we have

$$u i_k = v i_k \qquad (m + 1 \le k \le n),$$

then the relation

$$u a_{i_1} a_{i_2} \cdots a_{i_m} = v a_{i_1} a_{i_2} \cdots a_{i_m}$$

of the semigroup V_n is a consequence of the relations (Σ_1).

5°. **Theorem 1.** The system of relations (Σ_1) is a system of defining relations for the semigroup $V_n(2^0)$ with respect to the generating set $M_3(2^0)$.

Proof. By Lemmas 1 and 2, every word of the semigroup V_n with respect to M_3 is reduced, using the relations (Σ_1), to a word of the form

$$u a_{i_1} a_{i_2} \cdots a_{i_m} \qquad (u \in S_n,\ i_1 < i_2 < \cdots < i_m,\ 0 \le m \le n).$$

In view of what was said above, to prove the theorem it suffices to prove that every relation of the semigroup V_n of the form

$$ua_{i_1}a_{i_2}\cdots a_{i_m} = va_{j_1}a_{j_2}\cdots a_{j_{m_1}}, \tag{1}$$

where $u, v \in S_n$; $i_1 < i_2 < \cdots < i_m$; $j_1 < j_2 < \cdots < j_{m_1}$, $0 \le m, m_1 \le n$, is a consequence of the relations (Σ_1). Let $i_1, i_2, \ldots, i_n$ be a permutation of the numbers $1, 2, \ldots, n$. It is easy to see that if the relation (1) holds in the semigroup V, then

$$m = m_1, \qquad i_k = j_k, \qquad ui_r = vi_r \qquad (1 \le k \le m,\ m+1 \le r \le n).$$

It follows that the relation (1) has the form

$$ua_{i_1}a_{i_2}\cdots a_{i_m} = va_{i_1}a_{i_2}\cdots a_{i_m}, \tag{2}$$

where $ui_k = vi_k$ $(m+1 \le k \le n,\ 0 \le m \le n)$. If $m = 0, 1$, then u and v are identical, and therefore in this case relation (2) is a consequence of the relations (Σ_1). Let $2 \le m \le n$. Then, by Lemma 3, relation (2) is a consequence of the relations (Σ_1).

6°. Consider the following system of relations for the semigroup $W_n(2^0)$ with respect to the generating set $M_4(2^0)$:

1. Defining relations of the semigroup H_n with respect to $M_2(?)$.
2. Relations 2 and 3 from the system of relations (Σ_1).
3. $a_2a = a, \quad a_1a = aa_1a_2, \quad a_3a = aa_3$.
4. $aa_2 = a_2$.

$$(\Sigma_2)$$

7°. The following two lemmas can be proved.

Lemma 4. Let $u \in H_n$; $i \in \Omega$; and let $i_1, i_2, \ldots, i_m$ be the set of all those elements of Ω which under u are mapped to i. Then the relation

$$a_iu = ua_{i_1}a_{i_2}\cdots a_{i_m}$$

of the semigroup W_n is a consequence of the relations (Σ_2).

Lemma 5. Let $i_1, i_2, \ldots, i_n$ be a permutation of the numbers $1, 2, \ldots, n$, $1 \leqslant m \leqslant n$. If, for u, v in H_n, one has

$$ui_k = vi_k \qquad (m+1 \leqslant k \leqslant n),$$

then the relation

$$ua_{i_1}a_{i_2}\ldots a_{i_m} = va_{i_1}a_{i_2}\ldots a_{i_m}$$

of the semigroup W_n is a consequence of the relations (Σ_2).

8°. By Lemmas 2, 4, 5, analogously to the proof of Theorem 1, the following theorem can be proved:

Theorem 2. The system of relations (Σ_2) is a system of defining relations of the semigroup W_n (2°) with respect to the generating set M_4 (2°).

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[1] N. N. Vorob' ev, *Scientific Notes of the Leningrad State Pedagogical Institute named after A. I. Herzen*, **89**, 61 (1958).
[2] A. Ya. Aizenshtat, *Mathematical Collection*, **45** (87), No. 3 (1958).

Note: Figure translations are in progress. See original paper for figures.