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# MATHEMATICS

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1960

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**Abstract**

**Full Text**

## MATHEMATICS

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### ON THE COEFFICIENT-STABILITY OF DIFFERENCE SCHEMES

The question of the stability of the solution of difference boundary-value problems with respect to the coefficients of difference schemes (of coefficient-stability) is considered. It is shown that a necessary and sufficient condition for the coefficient-stability of a canonical scheme is its conservativeness.

1. Consider on the interval  $0 \leq x \leq 1$  the class of boundary-value problems

$$L^{(p,q,f)}u \equiv \frac{d}{dx} \left[ \frac{1}{p(x)} \frac{du}{dx} \right] - q(x)u + f(x) = 0, \quad 0 < x < 1, \\ u(0) = \mu_1, \quad u(1) = \mu_2. \quad (1)$$

The coefficients of the equation belong to the class  $Q_0$  of piecewise-continuous functions and satisfy the conditions:

$$0 < K_1 \leq p(x) \leq K_2, \quad 0 \leq q(x) \leq K_2, \quad |f(x)| \leq K_2, \quad (2)$$

where  $K_1$  and  $K_2$  are positive constants.

Let  $S_N = \{x_0 = 0, x_1 = h, \dots, x_i = ih, \dots, x_N = Nh = 1\}$  be a uniform difference grid with step  $h = \frac{1}{N}$ , and let  $L_h^{(p,q,f)}y_i$  be a homogeneous three-point difference scheme corresponding to the operator

$$L_h^{(p,q,f)}y_i = \frac{1}{h^2} [(y_{i+1} - y_i)/B_i^h - (y_i - y_{i-1})/A_i^h] - D_i^h y_i + F_i^h; \quad (3)$$

$$A_i^h = A^h[\bar{p}_i(s)], \quad B_i^h = B^h[\bar{p}_i(s)], \quad -1 < s < 1, \quad \bar{p}_i(s) = p(x_i + sh);$$

$$D_i^h = D^h[q(x_i + sh)], \quad F_i^h = F^h[f(x_i + sh)], \quad -0.5 < s < 0.5.$$

The functionals  $A^h$ ,  $B^h$ ,  $D^h$ , and  $F^h$  satisfy the conditions  $A_1, A_2, A_3$  of work <sup>(1)</sup>, i.e., we consider the same initial class of difference schemes as in work <sup>(1)</sup>. It is assumed here that  $D^h$  and  $F^h$  are linear functionals.

2. If  $B_i^h = A_{i+1}^h$ , then the difference operator  $L_h$  is called **conservative**. A conservative operator can be written in the self-adjoint form

$$L_h y_i \equiv \frac{1}{h^2} \Delta(\nabla y_i / A_i^h) - D_i^h y_i + F_i^h, \quad \text{where } \Delta y_i = y_{i+1} - y_i, \quad \nabla y_i = y_i - y_{i-1}. \quad (3')$$

Let us note that the difference scheme (3) can be made conservative by multiplication by the multiplier

$$\mu_i = \prod_{s=1}^{i-1} (A_{s+1}^h / B_s^h). \quad (4)$$

As a result, we obtain a conservative, generally speaking, nonhomogeneous scheme.

3. The difference Green' s function  $G_{i,k}$  is defined by the conditions

$$L_h^{(p,q)} G_{i,k} = -\frac{\delta_{i,k}}{h}, \quad G_{0k} = G_{Nk} = 0, \quad \delta_{ik} = \begin{cases} 1, & i = k, \\ 0, & i \neq k. \end{cases} \quad (5)$$

The solution of the boundary-value problem

$$L_h^{(p,q)} z_i = -\varphi_i, \quad 0 < i < N, \quad z_0 = 0, \quad z_N = 0 \quad (6)$$

is given by the formula

$$z_i = \sum_{k=1}^{N-1} G_{ik} \varphi_k h. \quad (7)$$

The Green' s function  $G_{ik}$  satisfies the following "symmetry" condition:  $\mu_i G_{ik} = \mu_k G_{ki}$ , where  $\mu_i$  is given by formula (4). For the conservative operator  $L_h^{(p,q)}$ ,  $B_s^h = A_{s+1}^h$ ,  $\mu_i = 1$ , and we obtain the symmetry condition  $G_{ik} = G_{ki}$ .

**Lemma 1.** *If the coefficients  $p(x), q(x)$  of the class  $Q_0$  satisfy conditions (2), and  $L_h^{(p,q)}$  is the initial difference scheme of the form (3), then the difference Green' s function  $G_{ik}$ , defined by conditions (5), and its first difference quotients*

$$(G_{i,k+1} - G_{i,k})/h \quad (0 \leq i \leq N, \quad 0 \leq k \leq N-1);$$

$$(G_{i+1,k} - G_{i,k})/h \quad (0 \leq i \leq N-1, 0 \leq k \leq N)$$

are bounded in absolute value by a constant depending only on  $K_1, K_2$ .

4. In solving difference boundary-value problems it may turn out that, for one reason or another, the coefficients of the difference equations are determined inaccurately. However, it is desirable that under a small distortion of the coefficients the solution of the problem should change little.

Let  $y_i$  and  $\tilde{y}_i$  be solutions of the difference boundary-value problems

$$L_h^{(p,q,f)} y_i = 0, \quad 0 < i < N, \quad y_0 = \mu_1, \quad y_N = \mu_2, \quad (8)$$

$$\tilde{L}_h^{(p,q,f)} \tilde{y}_i = 0, \quad \tilde{y}_0 = \mu_1, \quad \tilde{y}_N = \mu_2,$$

$$\tilde{L}_h^{(p,q,f)} \tilde{y}_i = h^{-2} (\Delta y_i |\tilde{B}_i^h - \nabla y_i | \tilde{A}_i^h) - \tilde{D}_i^h \tilde{y}_i + \tilde{F}_i^h. \quad (9)$$

Here the coefficients of the equation are distorted either through distortion of the coefficients of the differential equation, or through inaccuracy in the computation of the functionals  $A^h, B^h, D^h$ , and  $F^h$ , or, finally, as a result of both of the indicated causes.

We shall say that the difference scheme (3) **satisfies the principle of coefficient-stability** if, from the conditions

$$\begin{aligned} \sum_{i=1}^{N-1} |\tilde{A}_i^h - A_i^h| h &= \rho(h), & \sum_{i=1}^{N-1} |\tilde{B}_i^h - B_i^h| h &= \rho(h), \\ \sum_{i=1}^{N-1} |\tilde{D}_i^h - D_i^h| h &= \rho(h), & \sum_{i=1}^{N-1} |\tilde{F}_i^h - F_i^h| h &= \rho(h), \end{aligned} \quad (10)$$

where  $\rho(h) \rightarrow 0$  as  $h \rightarrow 0$ , there follows convergence of the solution  $\tilde{y}_i$  of the difference boundary-value problem (9) to the solution  $u(x)$  of problem (1), i.e.,

$$|\tilde{y}_i - u(x_i)| \leq \rho_0(h) \rightarrow 0 \quad \text{as } h \rightarrow 0. \quad (11)$$

Hence, in particular, it follows that for a coefficient-stable scheme

$$|y_i - u(x_i)| \leq \rho_1(h), \quad |y_i - \tilde{y}_i| \leq \rho_2(h),$$

where  $\rho_1(h), \rho_2(h) \rightarrow 0$  as  $h \rightarrow 0$ .

If in conditions (10) and (11) one replaces them by

$$\rho(h) = O(h^n), \quad \rho_0(h) = O(h^n), \quad (12)$$

then we obtain the principle of coefficient-stability of  $n$ -th order.

**5.** We formulate a necessary condition for coefficient-stability.

Let  $p(x)$  have a discontinuity at the point  $\xi = x_n + \theta h$ ,  $0 \leq \theta \leq 1$ ,  $x_n = nh$ , so that  $p_\ell = p(\xi - 0) \neq p_r = p(\xi + 0)$ . Introduce the function  $\tilde{p}(x, h)$ , coinciding with  $p(x)$  everywhere except on the intervals  $(x_n, x_{n+1})$  and  $(x_{n+1}, x_{n+2})$ . Then for coefficient-stability of a scheme  $\dot{L}_h^{(p,q,f)}$  of the form (3), it is necessary that the condition

$$\tilde{B}_n^h \tilde{B}_{n+1}^h \tilde{B}_{n+2}^h / p_r - \tilde{A}_n^h \tilde{A}_{n+1}^h \tilde{A}_{n+2}^h / p_\ell = \rho(h) \rightarrow 0 \quad \text{as } h \rightarrow 0 \quad (13)$$

be satisfied.

It is not difficult to notice that the necessary condition for convergence in the class of discontinuous coefficients, obtained earlier in works <sup>(3,4)</sup>, is a consequence of the necessary condition of coefficient-stability (13) (for  $\tilde{p} \equiv p$ ).

**6. Lemma 2.** *Every conservative scheme  $\dot{L}_h^{(p,q,f)}$  from the original family of schemes satisfies the necessary condition of coefficient-stability (13).*

**Lemma 3.** *Let  $y_i, \tilde{y}_i$  be solutions of the boundary-value problems*

$$L_h y_i = 0, \quad y_0 = \mu_1, \quad y_N = \mu_2; \quad \tilde{L}_h \tilde{y}_i = 0, \quad \tilde{y}_0 = \mu_1, \quad \tilde{y}_N = \mu_2,$$

where  $L_h, \tilde{L}_h$  are conservative difference operators of the form (3'), whose coefficients satisfy the conditions

$$0 < K_1 \leq A_i^h \leq K_2, \quad 0 \leq D_i^h \leq K_2, \quad |F_i^h| \leq K_2. \quad (2')$$

Then the inequality holds

$$|y_i - \tilde{y}_i|_{(0 \leq i \leq N)} \leq C \left\{ \sum_{k=1}^N |\tilde{A}_k^h - A_k^h| h + \sum_{k=1}^{N-1} |\tilde{D}_k^h - D_k^h| h + \sum_{k=1}^{N-1} |\tilde{F}_k^h - F_k^h| h \right\}, \quad (14)$$

where  $C$  is a constant depending only on  $K_1$  and  $K_2$ .

An analogous lemma holds for problem (1).

Choosing as  $\tilde{L}_h$  the exact scheme  $\tilde{L}_h^{(p,q,f)}$  (see (2)), and as  $L_h$  a conservative scheme  $\dot{L}_h^{(p,q,f)}$  from the original family of schemes and relying on Lemma 3, it is not difficult to prove the theorem (cf. (1)).

**Theorem 1.** *If a conservative scheme  $\dot{L}_h^{(p,q,f)}$  from the original family has, in some class  $C_{m_k, m_q, m_f}$ ,  $n$ -th integral order of accuracy, then it has this same  $n$ -th order of accuracy for coefficients from the class  $C_{n-1}^{(1)}$ , i.e., for  $p \in C_{n-1}^{(1)}$ ,  $q \in C_{n-1}^{(1)}$ ,  $f \in C_{n-1}^{(1)}$ .*

7. We now consider the canonical scheme (see (1))

$$L_h^{(p,q,f)} y_i = h^{-2} [\Delta y_i / B_i - \nabla y_i / A_i] - D_i y_i + F_i, \quad (15)$$

whose functionals do not depend on  $h_3$ , and require that it satisfy the necessary condition (13).

**Theorem 2.** *If the canonical difference scheme (15) from the original family of schemes satisfies the necessary condition of coefficient-stability (13), then it is conservative, i.e.  $B_i = A_{i+1}$  or  $B[\psi(s)] = A[\psi(1+s)]$ .*

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\*  $C_m^\gamma$  ( $0 \leq \gamma \leq 1$ ) is the class of functions having on the segment  $[0, 1]$  a continuous derivative of order  $m$ , satisfying a Hölder condition of order  $\gamma$ .

Relying on Theorem 2 and Lemmas 2 and 3, one can verify that:

**Theorem 3.** *Every homogeneous conservative scheme from the original family of schemes satisfies the principle of coefficient-stability.*

As a result we arrive at the following basic theorem.

**Theorem 4.** *A necessary and sufficient condition for the coefficient-stability of the canonical scheme  $L_h^{(p,q,f)}$  is its conservativeness.*

**Theorem 5.** *Every conservative scheme  $L_h^{(p,q,f)}$  has first integral order of accuracy in the class  $Q_1^0$ .*

§ 8. Let us now require that the difference scheme  $L_h^{(p,q,f)}$  satisfy the necessary conditions of coefficient-stability of rank 2.

**Theorem 6.** *There exists a unique canonical scheme ( "the best conservative scheme" ) having second integral order of accuracy in  $Q_1^{(1)}$  and satisfying the principle of coefficient-stability of rank 2; this scheme  $L_h^{(p,q,f)}$  is conservative and is determined by means of the functionals*

$$A[\psi] = \int_{-1}^0 \psi(s) ds, \quad D[\psi] = F[\psi] = \int_{-0.5}^{0.5} \psi(s) ds. \quad (16)$$

We note that in the proof of this theorem, in particular, Lemma 1 is used.

§ 9. Replacing the integral defining  $A[\psi]$  in formula (16) by a scheme based on some quadrature formula, we obtain, instead of the best canonical scheme  $L_h^{(p)}$ , a noncanonical scheme

$$\tilde{L}_h^{(p)} y_i = \frac{1}{h^2} \Delta(\nabla y_i / \tilde{A}_i^h), \quad \text{where } \tilde{A}_i^h = \tilde{A}^{h_1}[p(x_i + sh)],$$

$$\tilde{A}^{h_1}[\psi(s)] = \sum_{j=1}^J a_j \psi(s_j), \quad s_j = -1 + jh_1, \quad h_1 = 1/J.$$

**Theorem 7.** *In order that the noncanonical scheme  $\tilde{L}_h^{(p)}$  defined above have second integral order of accuracy in the class  $Q_1^{(1)}$ , it is necessary and sufficient that  $h_1/h = O(1)$  as  $h \rightarrow 0$  ( $N \rightarrow \infty$ ).*

An analogous theorem also holds for the scheme  $\tilde{L}_h^{(p,q,f)} y_i = \tilde{L}_h^{(p)} y_i - \tilde{D}_i^h y_i + \tilde{F}_i^h$ , whose functionals  $\tilde{D}^{h_1}$  and  $\tilde{F}^{h_1}$  are computed by analogy with the functional  $\tilde{A}^{h_1}$ .

§ 10. In work [1] we consider an asymptotic expansion for the solution of a difference boundary-value problem in the case of discontinuous coefficients. If  $p(x)$ ,  $q(x)$ , and  $f(x)$  are functions of the class  $Q_2^{(0)}$ , and  $L_h^{(p,q,f)}$  is the best canonical scheme, then the solution of problem (8) can be represented in the form

$$y_i = u(x_i) + h^2 Y(x_i, h) + O(h^4),$$

where  $Y(x, h) = O(1)$  and is a function having no limit as  $h \rightarrow 0$ . Hence it follows that, in the case of discontinuous coefficients, the solution of the difference boundary-value problem (8) does not have asymptotics of second order as  $h \rightarrow 0$ .

Received  
31 XII 1959

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\*  $Q_m^\gamma$  ( $0 \leq \gamma \leq 1$ ) is the class of functions having on  $[0, 1]$   $m$  piecewise-continuous derivatives, with the  $m$ -th derivative satisfying, in the intervals of its discontinuity, a Hölder condition of order  $\gamma$ .

*Note: Figure translations are in progress. See original paper for figures.*

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