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Abstract

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STRONG DIMENSION OF MAPPINGS AND THE DIMENSIONAL CHARACTERISTIC ASSOCIATED WITH IT FOR ARBITRARY METRIC SPACES*

(Presented by Academician P. S. Aleksandrov on 11 VI 1960)

Let a mapping f of a space** X into a space Y be given. Slightly modifying M. Katětov' s definition ([1]), we shall call the mapping f **strongly zero-dimensional *, if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that, if the diameter of an open set O of the space Y is less than δ, then its full inverse image $f^{-1}(O)$ is the sum of open pairwise disjoint sets H_α, each of which satisfies $\operatorname{diam} H_\alpha < \varepsilon$. Following Yu. Smirnov, we shall call the** strong dimension** of the mapping f the least of those natural numbers k such that the space X decomposes into the sum of $k+1$ sets X_i, on each of which the mapping f is strongly zero-dimensional. We shall agree to denote the strong dimension by $\operatorname{Dim} f$.

The principal aim of this note is the following two theorems, which give the desired characteristic of the dimension $\dim X$, one that is new even for spaces with a countable base:

Theorem 1. *Let f be a mapping of the space X into the space Y; then*

$$\dim X \leqslant \dim Y + \operatorname{Dim} f.$$

Theorem 2. *If $\dim X = n$, $k \leqslant n$, then there exists a bounded**** mapping f of the space X into the Euclidean space E^k such that $\operatorname{Dim} f = n-k$; moreover, the set of such mappings is everywhere dense in the space $C(X, E^k)$ of all bounded mappings of the space X into E^k.*

Corollary. *The dimension of the space X is at most n if and only if, for some (for every!) $k \leqslant n$, there exists a mapping f into E^k such that* $\operatorname{Dim} f \leqslant n-k$.

These theorems are a generalization of the known analogous theorems of W. Hurewicz ([3,4]) for compacta, where the dimension of mappings $\dim f$ is defined by means of the dimension of the full inverse images of points. For strongly zero-dimensional mappings these theorems were proved by M. Katětov ([1a]). It is known that Hurewicz' s first theorem in its direct formulation is not true even for spaces with a countable base: discarding in the known example

* Only the dimension of a space defined by means of coverings is considered.

** By a space we everywhere mean only a metric space, and by a mapping only a continuous mapping.

*** M. Katětov ([1]) called such mappings uniformly zero-dimensional. It seems to us correct to reserve this term for that concept of dimension of mappings which would give an analogous characteristic of uniform dimension in the sense of Yu. Smirnov ([2]).

**** A mapping f of a space X into a space Y is called **bounded** if the image $f(X)$ is a bounded set.

of a completely disconnected set of Knaster—Kuratowski ([5]) the vertex, we see that the projection π (from this point) of the remaining one-dimensional set onto the zero-dimensional Cantor set has dimension $\dim \pi = 0$. Moreover, Hurewicz ([6]) gave, for every n, a general method of constructing n-dimensional completely disconnected sets X^n, mapped one-to-one onto a Cantor set, and consequently also onto a line.

At the same time, the first Hurewicz theorem in its original form is valid for arbitrary spaces if only closed mappings are considered ([7],[8]). Nevertheless, P. S. Aleksandrov's tempting hypothesis of obtaining a dimension characteristic by replacing, in Hurewicz' s theorems, arbitrary mappings by closed ones was refuted by A. H. Stone ([9]), who constructed an example of a plane one-dimensional set for which there exists no closed zero-dimensional mapping onto a line. We now turn to the exposition of the paper.

Theorem A. *Let a mapping f of a space X into a space Y with a countable base be given; then, in the space X, any of its subsets M can be enclosed in a set M_0 of type G_δ, on which the mapping f has the same strong dimension as on M:* $\mathrm{Dim}_{M_0} f = \mathrm{Dim}_M f$.

Proof. We may assume that $\mathrm{Dim}_M f = 0$. For each number $\varepsilon_k = 1/2k$ there exists a number $\delta_k > 0$ such that the condition of strong zero-dimensionality is fulfilled on M. Consider the spherical neighborhoods $O(c_i, \delta_k)$ of the points c_i of some countable dense subset C in Y. Let

$$\Gamma_{ik} = f^{-1}(O(c_i, \delta_k))$$

and

$$H_{ik} = M \cap \Gamma_{ik}.$$

There are decompositions

$$H_{ik} = \bigcup_\alpha H_{ik}^\alpha,$$

where the H^{α}_{ik} are open in M,

$$\operatorname{diam} H^{\alpha}_{ik} < \frac{1}{2k}, \qquad H^{\alpha}_{ik} \cap H^{\beta}_{ik} = \emptyset$$

(if $\alpha \neq \beta$), where $\emptyset$ is the empty set. For any fixed i and k, by Yu. Smirnov' s lemma ([10]), the sets H^{α}_{ik} open in M can be extended to sets Γ^{α}_{ik} open in X so that

$$\operatorname{diam} \Gamma^{\alpha}_{ik} \leq \frac{1}{k}, \qquad \Gamma^{\alpha}_{ik} \cap \Gamma^{\beta}_{ik} = \emptyset, \quad \text{if } \alpha \neq \beta,$$

and so that

$$M \cap \Gamma_{ik} = \bigcup_{\alpha} \Gamma^{\alpha}_{ik} \subseteq \Gamma_{ik}.$$

The sets

$$V_{ik} = \Gamma_{ik} \setminus \bigcup_{\alpha} \Gamma^{\alpha}_{ik},$$

and together with them also the set

$$V = \bigcup_{i,k} V_{ik}$$

are of type F_{σ}. The set

$$M_0 = X \setminus V$$

is the desired one.

Theorem B*. *The projection π of Euclidean space E^n onto its subspace E^k has strong dimension* $\operatorname{Dim} \pi = n - k$.

Proof. Decompose the space E^{n-k} into the sum of $n-k+1$ zero-dimensional sets N_j. On each of the sets $E^k \times N_j$ the mapping π is strongly zero-dimensional, and

$$E^n = \bigcup_{j} (E^k \times N_j).$$

The theorem is proved.

Lemma 1. *If, in the superposition fg of two mappings, one of them is strongly zero-dimensional, then*

$$\operatorname{Dim} fg = \operatorname{Dim} f + \operatorname{Dim} g.$$

The proof is easily reduced to the simple case of strong zero-dimensionality of both mappings.

Lemma 2. *For every mapping g of a space X into a space Y one can find a mapping φ of the space X into the space $E^{\dim Y}$ such that*

$$\operatorname{Dim} \varphi = \operatorname{Dim} g.$$

Proof. By a theorem of M. Katětov ([16]), there exists **a strongly zero-dimensional mapping f of the space Y into $E^{\dim Y}$. The superposition fg is the required one.

* True, of course, is the following more general theorem:

Theorem B′. *Let the product $X \times Y$ of spaces X and Y be given; then for the projection π onto the space X we have* $\mathrm{Dim}\,\pi = \dim Y$.

The proof is carried out in exactly the same way with the aid of Katětov' s theorem ([1a]) on the possibility of decomposing an n-dimensional space into the sum of $n+1$ zero-dimensional sets.

** Even in the case when $\dim Y = \infty$. By E^∞ one should, of course, understand Hilbert space.

Proof of Theorem 1. By the preceding, in the condition of the theorem one may replace* the space Y by E^n, where $n = \dim Y$. The space E^n is easily represented as the sum of $n+1$ zero-dimensional sets N_i in such a way that every sum of the form $\bigcup_{i=0}^{p} N_i$ is of type F_σ. Let $N_i' = f^{-1}(N_i)$, and let $k = \mathrm{Dim}\, f$. By Theorem A, the space X can be represented as the sum of $k+1$ sets X_j of type G_δ, on each of which $\mathrm{Dim}_{X_j} f = 0$. Put $X_j' = X_j \setminus \bigcup_{i>j} X_i$. Then every sum of the form $\bigcup_{j=0}^{q} X_j'$ is of type F_σ. Finally, let $H_{ij} = N_i' \cap X_j'$. Since $\dim N_i = 0$ and $\mathrm{Dim}\, X_j' f = 0$, each of the sets H_{ij} has a base decomposing into the sum of a countable number of open coverings of multiplicity 1. By Morita' s theorem ([8]), then $\dim H_{ij} = 0$ for all i and j. One can show that under our assumptions every term H_{ij} of the sum $D_l = \bigcup_{i+j=l} H_{ij}$ is in it of type F_σ.

Therefore $\dim D_l = 0$. But $X = \bigcup_{l=0}^{n+k} D_l$.

Hence, by a theorem of Yu. Smirnov ([11]), $\dim X \leq n+k$, as was required to prove.

Proof of Theorem 2. Let $k \leq n = \dim X$. By a theorem of M. Katetov ([1]), there exists a strongly zero-dimensional mapping φ of the space X into E^n (even for $n = \infty$)**. For the superposition $\pi\varphi$, where π is the projection of the space E^n onto the subspace E^k, by Theorem 1 we have $\dim \pi\varphi \geq n-k$. At the same time $\dim \pi\varphi \leq n-k$ by Theorem A and Lemma 1. It is known ([12]) that the set of all such superpositions is dense in $C(X, E^k)$. The theorem is proved.

Remark. We have obtained a purely metric characteristic of the topological notion of dimension. Therefore, as a consequence, one can formulate a necessary condition for all metrics of the space X, and a sufficient condition for some metric of the space X.

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* This is necessary for the time being, since the space Y need not possess a countable base.

** In this case Theorem 2 must be formulated as follows:

Theorem 2′. If $k \leq n = \dim X$, then there exists a bounded mapping f of the space X into the space E^{n-k} such that $\operatorname{Dim} f = k$; moreover, the set of all such mappings is dense in the space $C(X, E^{n-k})$ ($\infty - k = \infty$ for every finite k. $\infty - \infty$ may mean any natural number and even ∞).

Note: Figure translations are in progress. See original paper for figures.