Fluid Mechanics

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Abstract

Full Text

Fluid Mechanics

A. S. Monin

On Lagrangian Characteristics of Turbulence

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1. In the theoretical study of turbulent diffusion (i.e., the spreading of admixtures in a turbulent medium as a result of their transport by moving fluid particles) and of other similar phenomena, it is more convenient (see below formulas (17)–(18)) to use not the Eulerian but the Lagrangian method of describing fluid motions, in which the principal characteristic of the fluid flow (we shall consider only an **incompressible** fluid) is the function $\mathbf{x}(t \mid \mathbf{x}_0, t_0)$, which determines, for any instant $t > t_0$, the coordinates (in the fixed reference frame S_0) of fluid particles that at the initial instant $t = t_0$ were at the points $\mathbf{x}_0$. Once this function is known, the velocities of the fluid particles are also known,

$$\mathbf{v}(t \mid \mathbf{x}_0, t_0) = \frac{\partial}{\partial t}\mathbf{x}(t \mid \mathbf{x}_0, t_0) = \mathbf{u}[\mathbf{x}(t \mid \mathbf{x}_0, t_0), t], \qquad (1)$$

where $\mathbf{u}(\mathbf{x}, t)$ is the Eulerian velocity field.

2. In the case of **small-scale** turbulence (whose external scale is small in comparison with the scale of observation), the values of the functions $\mathbf{x}$, $\mathbf{v}$, $\mathbf{u}$ entering (1) are statistically stable, so that $\mathbf{x}$, $\mathbf{v}$, $\mathbf{u}$ may be regarded as **random functions**; in particular, for the values of the coordinates $\mathbf{x}$ and velocities $\mathbf{v}$ of any finite number of fluid particles at any finite set of instants of time there exist probability distributions. The collection of all possible such distributions is a complete statistical description of turbulence in Lagrangian variables.

3. Lagrangian characteristics of small-scale turbulence were first introduced into the theory by G. Taylor (^{1}), who obtained (for the case of isotropic turbulence) a formula expressing the variance of the coordinate of a fluid particle $\mathbf{x}(t \mid \mathbf{x}_0, t_0)$ in terms of the Lagrangian correlation function of the velocity. For small $\tau = t - t_0$, the increment $\mathbf{x}(t \mid \mathbf{x}_0, t_0) - \mathbf{x}_0$ is approximately equal to $\mathbf{u}(\mathbf{x}_0, t_0)\tau$, and its variance grows as τ^2. The increments of the function $\mathbf{x}(t \mid \mathbf{x}_0, t_0)$ over nonoverlapping time intervals τ, large in comparison with the external time scale of the turbulence, are statistically independent, so that the sequence $\mathbf{x}_n = \mathbf{x}(n\tau \mid \mathbf{x}_0, 0)$ may be regarded as Markovian. The well-known semiempirical equation of

turbulent diffusion has meaning only as a continuum approximation of the corresponding difference equation for the indicated Markov sequence. It follows from Taylor' s formula that, in the case of homogeneous turbulence, for sufficiently large τ the variance of the increment $\mathbf{x}(t \mid \mathbf{x}_0, t_0) - \mathbf{x}_0$ is asymptotically proportional to τ; in the presence of a gradient of the Eulerian mean velocity this result is sharply distorted (^{2}).

4. The probability distribution for the random variable $\mathbf{v}(\mathbf{x}_1, t_1 \mid \mathbf{x}_0, t_0)$, defined as $\mathbf{v}(t_1, \mathbf{x}_0, t_0)$ for fixed $\mathbf{x}(t_1 \mid \mathbf{x}_0, t_0) = \mathbf{x}_1$, does not coincide with the distribution for the random variable $\mathbf{u}(\mathbf{x}_1, t_1)$, since the values of the latter are the velocities of **all possible** fluid particles that are at the point $\mathbf{x}_1$ at the instant t_1, whereas the values of $\mathbf{v}(\mathbf{x}_1, t_1 \mid \mathbf{x}_0, t_0)$ are the velocities only of those among the indicated fluid particles,

which at the instant t_0 were located at the point $\mathbf{x}_0$. However, for sufficiently large values of $t_1 - t_0$ (in comparison with the external time scale of the turbulence), the random quantities $\mathbf{v}(\mathbf{x}_1, t_1 \mid \mathbf{x}_0, t_0)$ and $\mathbf{u}(\mathbf{x}_1, t_1)$ may be regarded as equivalent.

5. By the theorem on total probability,

$$\begin{aligned} &\varphi(\mathbf{v}_1, \dots, \mathbf{v}_n;\ t_1, \dots, t_n \mid \mathbf{x}_{10}, \dots, \mathbf{x}_{n0};\ t_0) = \\ &= \int \varphi_1(\mathbf{v}_1, \dots, \mathbf{v}_n \mid \mathbf{x}_1, \dots, \mathbf{x}_n;\ t_1, \dots, t_n;\ \mathbf{x}_{10}, \dots, \mathbf{x}_{n0};\ t_0) \times \\ &\quad \times p(\mathbf{x}_1, \dots, \mathbf{x}_n;\ t_1, \dots, t_n \mid \mathbf{x}_{10}, \dots, \mathbf{x}_{n0};\ t_0)\, d\mathbf{x}_1 \cdots d\mathbf{x}_n, \end{aligned} \tag{2}$$

where φ and p are, respectively, the probability densities for the velocities $\mathbf{v}(t_1 \mid \mathbf{x}_{10}, t_0), \dots, \mathbf{v}(t_n \mid \mathbf{x}_{n0}, t_0)$ and the coordinates $\mathbf{x}(t_1 \mid \mathbf{x}_{10}, t_0), \dots, \mathbf{x}(t_n \mid \mathbf{x}_{n0}, t_0)$ of a finite number n of fluid particles, while φ_1 is the probability density for $\mathbf{v}(\mathbf{x}_1, t_1 \mid \mathbf{x}_{10}, t_0), \dots, \mathbf{v}(\mathbf{x}_n, t_n \mid \mathbf{x}_{n0}, t_0)$. If all the intervals $t_1 - t_0, \dots, t_n - t_0$ are sufficiently large, then φ_1 may be identified with the probability density for the Eulerian characteristics $\mathbf{v}_1 = \mathbf{u}(\mathbf{x}_1, t_1), \dots, \mathbf{v}_n = \mathbf{u}(\mathbf{x}_n, t_n)$.

One special case of formula (2) is the relation

$$\overline{\mathbf{v}(t \mid \mathbf{x}_0, t_0)} = \int \overline{\mathbf{v}(\mathbf{x}, t \mid \mathbf{x}_0, t_0)}\, p(\mathbf{x}, t \mid \mathbf{x}_0, t_0)\, d\mathbf{x}, \tag{3}$$

which determines the mean value of the velocity of a fluid particle (the bar above denotes mathematical expectation). For sufficiently large $t - t_0$, here one may put $\overline{\mathbf{v}(\mathbf{x}, t \mid \mathbf{x}_0, t_0)} \simeq \overline{\mathbf{u}(\mathbf{x}, t)}$.

6. In the case of stationary homogeneous turbulence, the velocity of a fixed fluid particle is a stationary random function of time. This is not true for $\mathbf{x}(t \mid \mathbf{x}_0, t_0)$, but the probability density P for $\mathbf{r} = \mathbf{x}(t_1 \mid \mathbf{x}_0, t_0) - \mathbf{x}(t_2 \mid \mathbf{x}_0, t_0)$ depends only on $\tau = t_1 - t_2$. For $t_1 > t_2$ and sufficiently large $t_2 - t_0$, (2) gives

$$B_{ij}(\tau) = \int b_{ij}(\mathbf{r}, \tau) P(\mathbf{r}, \tau)\, d\mathbf{r}, \tag{4}$$

where B_{ij} is the Lagrangian correlation function of the velocity, and $b_{ij} = \overline{v_i(\mathbf{r}, \tau \mid 0, 0) u_j(0, 0)}$.

7. The asymptotic laws indicated in §§ 3-6, valid for sufficiently large time intervals, have real significance only in the case of small-scale turbulence. In a number of problems, atmospheric turbulence cannot be regarded as small-scale; in this case only local hydrodynamic characteristics, but not the values $\mathbf{x}, \mathbf{v}, \mathbf{u}$, can be considered statistically stable.

Let the initial velocity $\mathbf{v}(t_0 \mid \mathbf{x}_0, t_0)$ of a fluid particle located at the instant t_0 at the point $\mathbf{x}_0$ have the fixed value $\mathbf{v}_0$. Introduce an inertial frame of reference S_1, moving with velocity $\mathbf{v}_0$; the coordinates in this frame will be the quantities $\mathbf{y} = \mathbf{x} - \mathbf{x}_0 - \mathbf{v}_0 \tau$, $\tau = t - t_0$. The relative (with respect to the frame S_1) coordinates $\mathbf{y}(\tau \mid \mathbf{y}_1)$ and velocities $\mathbf{v}(\tau \mid \mathbf{y}_1)$ of fluid particles which at the initial instant $\tau = 0$ were at the points $\mathbf{y}_1$ will be local Lagrangian characteristics of the flow, provided only that the values $|\mathbf{y}_1|$ and τ are sufficiently small in comparison with the external scales of the turbulence.

8. Consider, for fixed $\mathbf{y}_1$ and τ, the probability distribution for the local characteristics $\mathbf{y}(\tau \mid \mathbf{y}_1)$, $\mathbf{v}(\tau \mid \mathbf{y}_1)$. According to A. N. Kolmogorov' s theory of locally isotropic turbulence (^{3}), this distribution:

A. Does not depend on $t_0, \mathbf{x}_0, \mathbf{v}_0$.

B. Depends only on $\mathbf{y}_1$, τ, and the parameters ε (the rate of dissipation of turbulent energy) and ν (the kinematic viscosity).

The probability density of this distribution may be written in the form

$$f(\mathbf{y}, \mathbf{v}, \tau \mid \mathbf{y}_1) = \int \psi(\mathbf{y}, \mathbf{v}, \tau \mid \mathbf{y}_1, \mathbf{v}_1) \varphi(\mathbf{v}_1 \mid \mathbf{y}_1)\, d\mathbf{v}_1, \tag{5}$$

where φ is the probability density for the Eulerian velocity difference $\mathbf{v}_1 = \mathbf{u}(\mathbf{x}_0 + \mathbf{y}_1, t_0) - \mathbf{u}(\mathbf{x}_0, t_0)$, and ψ is the probability density for $\mathbf{y}(\tau \mid \mathbf{y}_1)$, $\mathbf{v}(\tau \mid \mathbf{y}_1)$ with fixed $\mathbf{v}(0 \mid \mathbf{y}_1) = \mathbf{v}_1$, which, in consequence of condition A, has the form

$$\psi(\mathbf{y}, \mathbf{v}, \tau \mid \mathbf{y}_1, \mathbf{v}_1) = \psi_0(\mathbf{y} - \mathbf{y}_1 - \mathbf{v}_1 \tau, \mathbf{v} - \mathbf{v}_1, \tau) \tag{6}$$

(Galilean invariance).

9. Let us dwell on the form of the probability density $\psi_0(\mathbf{y}, \mathbf{v}, \tau)$. In consequence of condition B,

$$\psi_0(\mathbf{y}, \mathbf{v}, \tau) = (\lambda v_\lambda)^{-3} \Psi\left(\frac{\mathbf{y}}{\lambda}, \frac{\mathbf{y} \cdot \mathbf{v}}{\lambda v_\lambda}, \frac{\mathbf{v}}{v_\lambda}, \frac{\tau}{\tau_\lambda}\right), \tag{7}$$

where $\lambda = \nu^{3/4}\varepsilon^{-1/4}$, $v_\lambda = \nu^{1/4}\varepsilon^{1/4}$, $\tau_\lambda = \nu^{1/2}\varepsilon^{-1/2}$ are the microscales of length, velocity, and time. According to (3), in the so-called **inertial range** (for $\tau \gg \tau_\lambda$) the dependence of ψ_0 on ν must become insignificant, and instead of (7) one obtains

$$\psi_0(\mathbf{y}, \mathbf{v}, \tau) = (\varepsilon\tau^2)^{-3}\Psi_0\left(\frac{y^2}{\varepsilon\tau^3}, \frac{\mathbf{y}\cdot\mathbf{v}}{\varepsilon\tau^2}, \frac{v^2}{\varepsilon\tau}\right). \tag{8}$$

The particular probability density for the relative coordinate of a fluid particle $\mathbf{y}(\tau \mid 0)$ then has the form

$$P(\mathbf{y}, \tau) = (\varepsilon\tau^3)^{-3/2}P_0\left(\frac{y^2}{\varepsilon\tau^3}\right). \tag{9}$$

This formula is of great importance for the description of **relative** turbulent diffusion.

10. Note that $\overline{\mathbf{y}(\tau \mid 0)} = \overline{\mathbf{v}(\tau \mid 0)} = 0$, and consider the form of the second moments of the distribution ψ_0. One can prove the relations

$$\overline{v_i(\tau \mid 0)v_j(\tau \mid 0)} = D(\tau)\delta_{ij}; \qquad \overline{y_i(\tau \mid 0)v_j(\tau \mid 0)} = \frac{1}{2}\tau D(\tau)\delta_{ij}; \tag{10}$$

$$\overline{y_i(\tau \mid 0)y_j(\tau \mid 0)} = \delta_{ij}\int_0^\tau \tau D(\tau)\,d\tau.$$

Here $D(\tau)$ is the Lagrangian structure function of velocity. For very small τ we have $\mathbf{v}(\tau \mid 0) \simeq \mathbf{w}\tau$ and $\mathbf{y}(\tau \mid 0) \simeq \frac{1}{2}\mathbf{w}\tau^2$, where $\mathbf{w}$ is the initial acceleration of the fluid particle. In this case

$$\overline{v_i(\tau \mid 0)v_j(\tau \mid 0)} \simeq \overline{w^2}\tau^2\delta_{ij}; \qquad \overline{y_i(\tau \mid 0)v_j(\tau \mid 0)} \simeq \frac{1}{2}\overline{w^2}\tau^3\delta_{ij}; \tag{11}$$

$$\overline{y_i(\tau \mid 0)y_j(\tau \mid 0)} \simeq \frac{1}{4}\overline{w^2}\tau^4\delta_{ij},$$

and the correlation coefficient between $y_i(\tau \mid 0)$ and $v_i(\tau \mid 0)$ is equal to unity. In the inertial range one obtains

$$\overline{v_i(\tau \mid 0)v_j(\tau \mid 0)} = C\varepsilon\tau\delta_{ij}; \qquad \overline{y_i(\tau \mid 0)v_j(\tau \mid 0)} = \frac{1}{2}C\varepsilon\tau^2\delta_{ij}; \tag{12}$$

$$\overline{y_i(\tau \mid 0)y_j(\tau \mid 0)} = \frac{1}{3}C\varepsilon\tau^3\delta_{ij},$$

where C is a positive numerical constant. The correlation coefficient between $y_i(\tau \mid 0)$ and $v_i(\tau \mid 0)$ is then equal to $\sqrt{3}/2$. A. M. Obukhov$^{(4)}$ showed that, under the assumption of a **Markovian** character of the six-dimensional random function $\{\mathbf{y}(\tau \mid \mathbf{y}_1, \mathbf{v}_1), \mathbf{v}(\tau \mid \mathbf{y}_1, \mathbf{v}_1)\}$ and the condition of Galilean invariance (6), the distribution ψ_0 proves to be normal with second moments (12).

11. Sometimes it is more convenient to use not the inertial reference system S_1, but a non-inertial reference system S_2 moving together with a **fixed** fluid particle (which at the moment t_0 was at the point $\mathbf{x}_0$). The local Lagrangian characteristics will then be the coordinates of fluid particles in the system S_2 (i.e. the quantities $\mathbf{l}(\tau \mid \mathbf{l}_0) = \mathbf{x}(t_0 + \tau \mid \mathbf{x}_0 + \mathbf{l}_0, t_0) - \mathbf{x}(t_0 + \tau \mid \mathbf{x}_0, t_0)$, as well as the velocities of fluid particles in the system S_2, for sufficiently small $|\mathbf{l}_0|$ and τ.

12. As a consequence of local isotropy, the probability density for the quantity $\delta\mathbf{l} = \mathbf{l}(\tau \mid \mathbf{l}_0) - \mathbf{l}_0$ has the form

$$q(\delta\mathbf{l}, \tau \mid \mathbf{l}_0) = \lambda^{-3} Q\left(\frac{|\delta\mathbf{l}|}{\lambda}, \frac{\delta\mathbf{l}\cdot\mathbf{l}_0}{\lambda l_0}, \frac{l_0}{\lambda}, \frac{\tau}{\tau_\lambda}\right). \tag{13}$$

For $l_0 \gg l$, the dependence on ν in (13) becomes immaterial, and one obtains

$$q(\delta\mathbf{l}, \tau' \mid \mathbf{l}_0) = (\varepsilon\tau^3)^{-3/2} Q_0\left(\frac{|\delta\mathbf{l}|^2}{\varepsilon\tau^3}, \frac{\delta\mathbf{l}\cdot\mathbf{l}_0}{\varepsilon^{1/2}\tau^{3/2} l_0}, \frac{l_0^2}{\varepsilon\tau^3}\right). \tag{14}$$

There exists such a small τ_1 that for $\tau < \tau_1$ one may set $\delta\mathbf{l} \approx \delta\mathbf{u}\cdot\tau$, where $\delta\mathbf{u} = \mathbf{u}(\mathbf{x}_0 + \mathbf{l}_0, t_0) - \mathbf{u}(\mathbf{x}_0, t_0)$ is the Eulerian velocity difference. In this case

$$q(\delta\mathbf{l}, \tau \mid \mathbf{l}_0) \approx \tau^{-3} \varphi\left(\frac{\delta\mathbf{l}}{\tau}\middle|\, \mathbf{l}_0\right), \tag{15}$$

where $\varphi(\delta\mathbf{u} \mid \mathbf{l}_0)$ is the probability density for $\delta\mathbf{u}$. On the other hand, since the mean square distance between two fluid particles increases with time, there exists such a τ_2 that for $\tau > \tau_2$ one will have $\overline{|\mathbf{l}(\tau \mid \mathbf{l}_0)|^2} \gg l_0^2$, and the dependence of $q(\delta\mathbf{l}, \tau \mid \mathbf{l}_0)$ on $\mathbf{l}_0$ will become immaterial (according to the supposition of G. K. Batchelor^5, for small l_0 this occurs before the motions of the two particles become independent). In this case formulas (13) and (14) take the form

$$q(\delta\mathbf{l}, \tau) = \lambda^{-3} Q_1\left(\frac{|\delta\mathbf{l}|}{\lambda}, \frac{\tau}{\tau_\lambda}\right); \tag{13'}$$

$$q(\delta\mathbf{l}, \tau) = (\varepsilon\tau^3)^{-3/2} Q_{01}\left(\frac{|\delta\mathbf{l}|^2}{\varepsilon\tau^3}\right). \tag{14'}$$

Formula (14′), analogous to (9), is also obtained from (13′) for $\tau \gg \tau_\lambda$. Let us note that the quantities τ_1/τ_λ and τ_2/τ_λ are functions of l_0/λ, and the relation between τ_1, τ_2, and τ_λ is determined mainly by the magnitude of l_0.

13. The transport of an admixture by moving fluid particles in an incompressible fluid is described by the equation

$$\frac{\partial s}{\partial t} + \frac{\partial u_\alpha s}{\partial x_\alpha} = 0, \tag{16}$$

where $s(\mathbf{x}, t)$ is the volumetric concentration. The mathematical expectation of the solution of this equation, for a fixed initial concentration field $s_0(\mathbf{x})$, has the form

$$\overline{s(\mathbf{x}, t)} = \int p(\mathbf{x}, t \mid \mathbf{x}_0, t_0)\, s_0(\mathbf{x}_0)\, d\mathbf{x}_0, \tag{17}$$

where p is the probability density for the coordinate of the fluid particle $\mathbf{x}(t \mid \mathbf{x}_0, t_0)$. This formula is fundamental for the theory of turbulent diffusion*. Passing in equation (16) to the inertial frame of reference S_1, we obtain an analogue of formula (17) for **relative** diffusion:

$$\overline{s(\mathbf{y}, \tau)} = \int p(\mathbf{y}, \tau \mid \mathbf{y}_1)\, s_0(\mathbf{y}_1)\, d\mathbf{y}_1, \tag{18}$$

where p is obtained from function (5) by integration with respect to $\mathbf{v}$. An analogous formula is also valid in the frame of reference S_2.

Institute of Atmospheric Physics
Academy of Sciences of the USSR

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* Accounting for molecular diffusion reduces, roughly speaking, to Gaussian smoothing with respect to $\mathbf{x}$ of the function p in (13), which can be significant only if the dispersion of the distribution p is not too large in comparison with $2\chi(t-t_0)$, where χ is the coefficient of molecular diffusion. For $t-t_0 \gg 2\chi/{u'}^2$, molecular diffusion may be neglected in (13).