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K. K. GOLOVKIN

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Abstract

Full Text

MATHEMATICS

K. K. GOLOVKIN

ON CONDITIONS FOR THE SMOOTHNESS OF FUNCTIONS

(Presented by Academician V. I. Smirnov, 1 VI 1960)

1. We shall call any bounded measurable function $K_n(x)$ satisfying the conditions

$$\int_{-\infty}^{\infty} K_n(x)x^k dx = n! \delta_n^k, \quad \int_{-\infty}^{\infty} |K_n(x)| |x|^k dx < \infty, \quad 0 \leq k \leq n. \quad (1)$$

a kernel of order n .

Denote by $K_n^{(h)}u(x)$ the integral operator

$$K_n^{(h)}u(x) = \frac{1}{h} \int_{-\infty}^{\infty} K_n\left(\frac{y-x}{h}\right) u(y) dy. \quad (2)$$

Lemma 1. If $u(x)$ has continuous bounded derivatives up to order n , then for every kernel $K_n(x)$ of order n the equality

$$\lim_{h \rightarrow 0} \frac{1}{h^n} K_n^{(h)}u(x) = u^{(n)}(x) \quad (3)$$

holds.

We shall study classes of summable functions $u(x)$ satisfying the condition

$$\sup_x |K_n^{(h)}u(x)| = O(h^\lambda), \quad \lambda \leq n, \quad (4)$$

or, if $\lambda < n$, the analogous condition in which O is replaced by o . These classes become normed spaces, which we shall denote respectively by $K_n^{(\lambda)}$ or $k_n^{(\lambda)}$, if the norm is defined as

$$\sup_{x,h} \frac{1}{h^\lambda} |K_n^{(h)}u(x)|.$$

Along with this, results will be formulated for spaces of functions $G_{n,p}^{(\lambda)}$ ($\lambda \leq n$) and $g_{n,p}^{(\lambda)}$ ($\lambda < n$), having finite norm

$$\sup_h \frac{1}{h^\lambda} \|K_n^{(h)} u(x)\|_{L_p},$$

where for functions from $g_{n,p}^{(\lambda)}$ the additional condition must hold:

$$\lim_{h \rightarrow 0} \frac{1}{h^\lambda} \|K_n^{(h)} u(x)\|_{L_p} = 0.$$

It is proved elementarily:

Lemma 2. The spaces $k_n^{(\lambda)}$ and $g_{n,p}^{(\lambda)}$ coincide with the closure, in the corresponding norms, of the set of infinitely differentiable functions.

The norm in $K_n^{(\lambda)}$ ($k_n^{(\lambda)}$) is a homogeneous functional having dimension equal to $-\lambda$ and differential order λ (1). Therefore it is natural to compare these spaces with the spaces

$C^{(\lambda)}$ ($c^{(\lambda)}$) of sufficiently smooth functions, the norm in which is defined as:

$$\sup_{x,h} \left| \frac{u^{([\lambda])}(x+h) - u^{([\lambda])}(x)}{h^{\lambda-[\lambda]}} \right| \quad \text{for fractional } \lambda;$$

$$\sup_{x,h} \left| \frac{u^{(\lambda-1)}(x+h) - 2u^{(\lambda-1)}(x) + u^{(\lambda-1)}(x-h)}{h} \right| \quad \text{for integral } \lambda,$$

and the space $c^{(\lambda)}$ coincides with the closure, in these norms, of the infinitely differentiable functions.

Let us clarify some properties of kernels of order n . Let $K_n(x)$ be such a kernel. Then the differences $\frac{1}{2^{n+1}} K_n(x/2) - K_n(x)$ and $K_n(x+1) - K_n(x)$ will, after the corresponding normalization, be kernels of order higher than n . The following proposition is valid, having the character of the converse to what has been stated.

For every kernel $K_{n+1}(x)$ of order $n+1$ there exist absolutely integrable kernels $A_1(x)$ and $A_2(x)$ and a kernel $K_n(x)$ of order n such that

$$\frac{1}{2^{n+1}} K_n\left(\frac{x}{2}\right) - K_n(x) = \int_{-\infty}^{\infty} A_1(x-y) K_{n+1}(y) dy, \quad (5)$$

$$K_n(x+1) - K_n(x) = \int_{-\infty}^{\infty} A_2(x-y) K_{n+1}(y) dy, \quad (6)$$

$$\int_{-\infty}^{\infty} |A_1(y)| dy + \int_{-\infty}^{\infty} |A_2(y)| dy = A. \quad (7)$$

The problem of finding $A_1(x)$, $A_2(x)$, and $K_n(x)$ is easily solved after passing to Fourier transforms.

Lemma 3. If $u(x) \in K_n^{(\lambda)}$ ($k_n^{(\lambda)}$) for $n > \lambda$, then for every integer $m \in (\lambda, n]$ there exists a kernel of order m such that $u(x)$ belongs to the corresponding $K_m^{(\lambda)}$ ($k_m^{(\lambda)}$).

Obviously, it suffices to prove the lemma for $m = n - 1$ in the case when this quantity is greater than λ . Take a kernel $K_m(x)$ satisfying identity (5), and convolve both sides of (5) with $u(xh)$. Dividing the result by h^λ , we easily arrive at the inequality

$$\|u\|_{K_m^{(\lambda)}} \leq \frac{1}{2^{m-\lambda}} \|u\|_{K_n^{(\lambda)}} + A \|u\|_{K_n^{(\lambda)}} \leq \frac{A \|u\|_{K_n^{(\lambda)}}}{1 - (1/2)^{m-\lambda}},$$

which proves the lemma.

Theorem 1. In order that the summable function $u(x)$ belong to $C^{(\lambda)}$ ($c^{(\lambda)}$), it is necessary and sufficient that it belong to some space $K_l^{(\lambda)}$ ($k_l^{(\lambda)}$) with $l > \lambda$.

The necessity of the condition of the theorem is almost trivial. Let us outline the proof of sufficiency, assuming λ fractional and beginning with the spaces $c^{(\lambda)}$ and $k_l^{(\lambda)}$. By Lemma 3, we may assume $l = [\lambda] + 1$. By Lemma 2, it will be sufficient to establish an estimate of the norm in $c^{(\lambda)}$ in terms of the norm in $k_l^{(\lambda)}$ for $[\lambda]$ -times differentiable functions. From the given kernel $K_{[\lambda]+1}(x)$ we construct a kernel $K_{[\lambda]}(x)$ satisfying (5) and (6). Next we write formula (3), putting $n = [\lambda]$ in it, and on the basis of (5) estimate the order of convergence in it, which turns out to be equal to $o(h^{\lambda-[\lambda]})$. After this the difference $u^{[\lambda]}(x+h) - u^{[\lambda]}(x)$ is replaced approximately by the difference $K_{[\lambda]}^{(h)}u(x+h) - K_{[\lambda]}^{(h)}u(x)$, which is estimated on the basis of (6) and also turns out to be of order $o(h^{\lambda-[\lambda]})$. This completes the proof of the required estimate. Now in

in the case of the spaces $C^{(\lambda)}$ and $K_l^{(\lambda)}$ we may regard as already proved the existence for $u(x)$ of a continuous derivative of order $[\lambda]$, since the embedding $K_l^{(\lambda)} \supset k_l^{(\lambda')}$ holds for every $\lambda' < \lambda$. Then all the rest of the proof goes through without change. The case of integral λ requires a certain modification of the proof, which is still based only on Lemmas 1–3 and formulas (5) and (6).

Theorem 2. For a summable function $u(x)$, the conditions $u(x) \in K_l^{(l)}$ and $u^{(l-1)}(x) \in \text{Lip } 1$ are equivalent.

Theorem 3. For a summable function $u(x)$, for any $n > r$ the conditions $u(x) \in G_{n,p}^{(r)}$ ($g_{n,p}^{(r)}$), $u(x) \in H_p^{(r)}$ ($h_p^{(r)}$) are equivalent.

Theorem 4. For a summable function $u(x)$, the conditions $u(x) \in G_{l,p}^{(l)}$, $u(x) \in W_p^{(l)}$ are equivalent.

In proving the last theorem we rely on the equivalence, established by A. A. Dezin ⁽²⁾, of the conditions

$$\|u(x+h) - u(x)\|_{L_p} = O(h)$$

and $u(x) \in W_p^{(1)}$.

Corollary. Let $\omega(x)$ be a kernel of zero order (an averaging kernel), whose moments from the first to the $(n-1)$ -st are equal to zero, while the n -th moment is different from zero. Construct the mean of the summable function with respect to this kernel, and let h be the averaging parameter. Then the smallness of order $O(h^\lambda)$, $o(h^\lambda)$ of the norm of the difference $u(x) - u_h(x)$ in L_p or C is equivalent to precisely that smoothness of $u(x)$ which is guaranteed by Theorems 1–4 under the condition that $u(x)$ belongs respectively to $G_{n,p}^{(\lambda)}$ or $K_n^{(\lambda)}$.

2. Let us write an elementary identity having the form of the Fourier transform of (5), when the role of kernels is played by certain linear combinations of δ -functions:

$$\frac{1}{2^n} (e^{2i\xi} - 1)^n - (e^{i\xi} - 1)^n = (e^{i\xi} - 1)^{n+1} \sum_{k=0}^{n-1} \frac{1}{2^{k+1}} (e^{i\xi} + 1)^k. \quad (8)$$

Identity (8) has as its consequence the coincidence of the coefficients of all powers of $e^{i\xi}$ in the right- and left-hand sides. We may therefore substitute in (8), in place of the quantities $e^{i\xi m}$, the values of a completely arbitrary function $u(x)$ at the points $x + hm$. This gives us

$$\Delta_h^{(n)} u(x) = \frac{1}{2^n} \Delta_{2h}^{(n)} u(x) - \sum_{l=0}^{n-1} \alpha_{l,n} \Delta_h^{(n+1)} u(x + hl), \quad (9)$$

where

$$\alpha_{l,n} = \sum_{k=l}^{n-1} \frac{1}{2^{k+1}} C_k^{(l)}, \quad \Delta_h^{(m)} u(x) = \sum_{k=0}^m (-1)^{m-k} C_m^{(k)} u(x + hk).$$

From (9) it obviously follows that

Lemma 4. If $\|u(x)\| < \infty$, then the conditions $\|\Delta_h^{(m)} u(x)\| = O(h^\lambda)$ ($o(h^\lambda)$) for all $m > \lambda$ are equivalent, where $\|\cdot\|$ is any functional norm satisfying the condition $\|u(x+h)\| = \|u(x)\|$ for all h .

From the lemma just proved there follows, in particular, Zigmund' s theorem ⁽³⁾ on the equivalence of the conditions

$$|\Delta_h^{(2)} u(x)| = O(h^\lambda)$$

and

$$|\Delta_h^{(1)}u(x)| = O(h^\lambda)$$

for $\lambda < 1$, which was proved in ⁽³⁾ by means of the constructive theory of functions. V. A. Solonnikov found an elementary proof of this fact, which influenced our work. With the aid of identity (9) and Lemma 4 one can prove the theorems stated below.

Theorem 5. For a bounded function $u(x)$, for any $n > \lambda$ the conditions

$$\sup_x |\Delta_h^{(n)}u(x)| = O(h^\lambda) \quad (o(h^\lambda)), \quad u(x) \in C^{(\lambda)} \quad (c^{(\lambda)})$$

are equivalent.

Theorem 6. For a bounded function $u(x)$, the conditions

$$\sup_x |\Delta_h^{(n)}u(x)| = O(h^n), \quad u^{(n-1)}(x) \in \text{Lip } 1$$

are equivalent.

Theorem 7. For a function $u(x) \in L_p$, for any $n > r$, the conditions

$$\|\Delta_h^{(n)}u(x)\|_{L_p} = O(h^r) \quad (o(h^r)), \quad u(x) \in H_p^{(r)} \quad (h_p^{(r)})$$

are equivalent.

Theorem 8. For a function $u(x) \in L_p$, the conditions

$$\|\Delta_h^{(l)}u(x)\|_{L_p} = O(h^l), \quad u(x) \in W_p^{(l)}$$

are equivalent.

We note that in Theorems 5 and 6 the measurability of $u(x)$ is not assumed. Theorem 5 follows from a more general result of S. N. Bernstein ⁽⁴⁾.

Leningrad Branch
of the V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

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