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Abstract

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ON THE FLOW OF A CONDUCTING GAS AROUND A PLATE WITH CURRENT

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When a conducting gas flows around magnetic inhomogeneities, regions free of gas (cavities) may arise near singularities of the magnetic field. At large magnetic Reynolds numbers, when the thickness of the magnetic boundary layer becomes small in comparison with the characteristic dimensions of the problem, one may assume that the magnetic field is zero outside the cavity. It is evident that at the boundary of the cavity the condition of mechanical equilibrium must be satisfied

$$p = \frac{H^2}{8\pi} \tag{1}$$

(p is the gas pressure, H is the magnetic-field strength). Such a model of flow processes was proposed by J. Burgers ([1]) and V. N. Zhigulev ([2]), who also solved the problem of the flow around a linear current; for the pressure they used the modified Newton formula

$$p = p_0 \sin^2 \theta, \tag{2}$$

which describes the pressure of an ideal gas at zero temperature. Here θ is the angle of inclination of the cavity boundary to the direction of the oncoming flow.

In the present note, in the same approximation, the plane problem of the flow of a conducting gas around a plate with current is considered.

§ 1. We shall assume that the direction of the oncoming flow at infinity coincides with the x-axis, that a plate of width L is inclined to the x-axis at an angle α, and that it is not in contact with the gas (Fig. 1). The condition under which the latter assumption is satisfied will be considered somewhat later. The magnetic field is described by the complex potential $w(z)$. $\operatorname{Im} w$ takes a constant value on the plate and on the boundary of the cavity. Let us make a cut in the z-plane along the line on which the plate is situated. We shall conformally map the region of the cavity with the cut onto the interior of the ring $\varepsilon < |\zeta| < 1$ in such a way that the boundary of the cavity is transformed into the circumference of unit radius, and the cut into the circumference of radius ε. Then

$$w = \frac{2iI}{c} \ln \zeta,$$

where I is the total current flowing in the plate. Equation (1), together with (2), takes the form

$$\sin^2 \theta \left| \frac{dz}{d\zeta} \right|^2 = \lambda^2, \qquad \lambda = \frac{I}{c\sqrt{2\pi p_0}}. \tag{3}$$

Integrating (3), we obtain

$$y = \lambda(\pi - \varphi), \qquad \varphi = \arg \zeta. \tag{4}$$

The function effecting the indicated conformal mapping has the form

$$z = \sum_{a=1}^{\infty} (a_n \zeta^n + b_n \zeta^{-n}) \tag{5}$$

and can now be found from the boundary conditions: for $|\zeta| = 1$ its imaginary part is determined by (4), while for $|\zeta| = \varepsilon$ its imaginary and real parts are connected by the relation $\mu x + y = 0$, $\mu = \operatorname{tg} \alpha$. The function (5) is, generally speaking, complex: $a_n = a_{1n} + i a_{2n}$, $b_n = b_{1n} + i b_{2n}$. Expanding (4) in a Fourier series, for the coefficients a_{kn} and b_{kn} we obtain the expressions

$$a_{1n} = \frac{2\lambda}{n} + b_{1n} = \frac{2\lambda}{n} \frac{\mu^2(1+\varepsilon^{2n}) + 1 - \varepsilon^{2n}}{\mu^2(1+\varepsilon^{2n})^2 + (1-\varepsilon^{2n})^2},$$

$$a_{2n} = -b_{2n} = \frac{4\lambda}{n} \frac{\mu \varepsilon^{2n}}{\mu^2(1+\varepsilon^{2n})^2 + (1-\varepsilon^{2n})^2}.$$

The parameter ε is determined by the ratio L/λ. To find it, it is necessary to compute the difference between the maximum and minimum values of $x(\varphi)$ or $y(\varphi)$ for $|\zeta| = \varepsilon$. Investigation of the series (5) in the general case is quite complicated. Let us dwell in more detail on certain special cases.

§ 2. In the case of longitudinal flow ($\alpha = 0$), the function (5) is real:

$$a_n = \frac{2\lambda}{n} + b_n = \frac{2\lambda}{n(1-\varepsilon^{2n})},$$

and the final results can be expressed in terms of elliptic functions.

Put

$$\varepsilon = q = \exp\left(-\frac{\pi K'}{K}\right),$$

where K, K' are complete elliptic integrals of the first kind. For $\zeta = e^{i\varphi}$, $x(\varphi)$, as is seen from (5), reaches its maximum value at $\varphi = 0$ and its minimum at $\varphi = \pi$. Taking the difference between these values and summing the series thereby obtained ([3], p. 310), we arrive at an equation determining the modulus k of the elliptic functions:

$$k^2 = 1 - e^{-2L/\lambda}.$$

Fig. 1

To determine the equation of the boundary, put in (5) $\zeta = e^{i\varphi}$ and express φ in terms of y. Using the expansion of theta functions in a Fourier series, the equation can be represented in the following form:

$$e^{-x/\lambda} = \frac{\pi}{K}\,(2qkk')^{-1/3}\vartheta_2^2\left(\frac{y}{2\pi\lambda}\right). \tag{6}$$

If the plate is placed so that its left end is at the origin of coordinates, then the contour of the cavity is situated inside the curve corresponding to the line current ($L = 0$); at infinity they coincide. For fixed current, the braking force acting on the plate tends to zero as L increases.

§ 3. For transverse flow ($\alpha = \pi/2$), the function (5) is also real:

$$a_n = \frac{2\lambda}{n(1+\varepsilon^{2n})}.$$

As in the preceding case, put $\varepsilon = q$. For $\zeta = e^{i\varphi}$, for $y(\varphi)$ from (5) we have ([3])

$$y(\varphi) = 2\lambda\left[\operatorname{am}\left(\frac{K\varphi}{\pi}\right) - \frac{\varphi}{2}\right]. \tag{7}$$

The values φ^* at which y reaches its maximum and minimum are determined by the condition

$$\operatorname{dn}\left(\frac{\varphi^* K}{\pi}\right) = \frac{\pi}{2K}. \tag{8}$$

Since $2k'K < \pi$, equation (8) always has only two roots for $0 < \varphi < 2\pi$. Taking the difference between the maximum and minimum values, with the aid of (7) and (8) we obtain an equation for k:

$$\frac{L}{\lambda} = 4\left[\tau - \frac{\pi F(\tau, k)}{2K}\right], \tag{9}$$

where

$$\tau = \arcsin \frac{\sqrt{1 - \left(\frac{\pi}{2K}\right)^2}}{k}$$

and $F(\tau, k)$ is an elliptic integral of the first kind.

The function appearing on the right-hand side of (9) varies monotonically from 0 to 2π as k varies from 0 to 1.

Thus, the solution of the problem is unique and exists for $L \leqslant 2\pi\lambda$. Physically this is connected with the fact that, for $L = 2\pi\lambda$, the plate comes into contact with the gas, and the model under consideration loses its meaning.

Putting $\zeta = e^{i\varphi}$ in (5) and expressing φ through y, the equation of the cavity boundary in this case can be represented in the form[3]

$$e^{-2x/\lambda} = \frac{k(1+k')\,\mathrm{sn}^2(K/2)}{q^{1/2}} \frac{\mathrm{cn}(Ky/\pi\lambda)}{\mathrm{dn}(Ky/\pi\lambda) + k'}. \tag{10}$$

When $L = 2\pi\lambda$, the cavity degenerates into a rectangular semi-infinite strip of width $2\pi\lambda$, symmetric with respect to the x-axis. An analogous picture will occur for an arbitrary angle of inclination of the plate when $L = L_{\mathrm{cr}} = 2\pi\lambda/\sin\alpha$: the plate comes into contact with the gas, as follows from (4), and the cavity is an oblique semi-infinite strip. Thus, the condition of applicability of the model under consideration is $L < L_{\mathrm{cr}}$.

Let us give the expression for the reaction of the gas on the plate in the case where $L \ll 2\pi\lambda$. The drag force acting per unit length coincides, in order of magnitude, with the expression given in (2): $X \simeq \pi\lambda p_0$. For the lift force we have the expression

$$Y \simeq \sin 2\alpha \left(\frac{L}{8\lambda}\right)^2 X.$$

The moment of the forces R about the origin is connected with the lift force by the simple relation $R = \lambda Y$.

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Note: Figure translations are in progress. See original paper for figures.