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Abstract

Full Text

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ON PRIME NUMBERS IN SHORT ARITHMETIC PROGRESSIONS

(Presented by Academician I. M. Vinogradov, 9 IV 1960)

Yu. V. Linnik proved ⁽¹⁾ the existence of a constant $c > 1$, independent of k, l , for which in the interval $1 < x < k^c$ there is at least one prime number belonging to any arithmetic progression $ku + l$, $(k, l) = 1$, $u = 0, 1, 2, \dots$. K. A. Rodosskii found ⁽²⁾ a simpler proof, in which Linnik's fundamental lemma was replaced by a weaker estimate. By the same means I proved ⁽³⁾ the existence of a prime number $p \equiv l \pmod{k}$ $((k, l) = 1)$ in the interval

$$(x, xk^c) \tag{1}$$

for any $x \geq 1$.

In the present note a scheme is given for the proof of the following theorem, which improves (1).

Theorem. There exist positive constants c, c' , independent of k and l , such that for any positive $\varepsilon \leq c$, for all $k \geq k_0(\varepsilon)$ and all $x \geq k^{c' \log(c/\varepsilon)}$, in the interval (x, xk^ε) there is a prime number $p \equiv l \pmod{k}$ $((k, l) = 1)$.

The function $k_0(\varepsilon)$ depends on Siegel's constant and therefore is still not known in explicit form.

In what follows $A, B, C, c_0, \dots, c_4, \alpha, \eta_0, a, b$ denote positive constants independent of k, l, ε . In the proof of the theorem the following lemmas play an important role.

Lemma 1. Let $N(\delta, T)$ be the number of zeros of all Dirichlet L -functions $L(s, \chi)$ with characters χ modulo k , belonging to the rectangle $(\sigma \geq 1 - \delta, |t| \leq T)$ of the plane of complex numbers $s = \sigma + it$. Then for all $\lambda \in [0, \log k]$ we have

$$N(\lambda \mid \log k, e^\lambda \mid \log k) < e^{C\lambda}, \tag{2}$$

where C does not depend on λ .

Lemma 2. For a suitable c_0 the half-plane $G[\sigma \geq 1 - c_0/\log k(|t| + 2)]$ is free of zeros of any function $L(s, \chi)$, with the possible exception of one function $L(s, \chi_1)$ (with real non-principal character χ_1), which has in G a simple zero $\beta_1 = 1 - \delta_1 < 1$. According to Siegel's theorem, $\delta_1 \geq c_1(\varepsilon')k^{-\varepsilon'}$ for any $\varepsilon' > 0$.

For suitable A and

$$\lambda_0 = A \log \frac{eA}{\delta_0 \log k}, \quad \text{where } \delta_0 = \begin{cases} \delta_1, & \text{if } \delta_1 \leq A/\log k, \\ A/\log k, & \text{if } \delta_1 > A/\log k, \end{cases}$$

the rectangle $(1 - \lambda_0/\log k \leq \sigma \leq 1, |t| \leq k)$ contains no zeros $\rho \neq \beta_1$ of any L -function with characters mod k .

The proof of the theorem is, in its general outline, similar to Rodoskii's proof (2). The identity used is

$$\sum_{n=2}^{\infty} \chi(n) \Lambda(n) n^{-s} \exp\left(-\frac{\log^2 n/x}{4y}\right) = i \sqrt{\frac{y}{\pi}} \int_{2-i\infty}^{2+i\infty} \frac{L'}{L}(w, \chi) x^{w-s} e^{(w-s)^2 y} dw, \quad (3)$$

where x, y are arbitrary positive numbers; $\Lambda(n) = \log p$ if n is a power of the prime p , and is equal to 0 for the remaining n ; in what follows it is assumed that $s = -1/2$. After multiplying (3) by $\bar{\chi}(l)$ and summing over χ , the contour of integration is shifted to $\operatorname{Re} w = -1/2$. We obtain

$$\begin{aligned} \Phi(x, y; k, l) &= \varphi(k) \sum_{\substack{n=2 \\ n \equiv l \pmod{k}}}^{\infty} \Lambda(n) \sqrt{n} \exp\left(-\frac{\log^2 n/x}{4y}\right) = \\ &= 2\sqrt{\pi y} x^{3/2} e^{9y/4} (1 - S) + O(k \log k), \end{aligned} \quad (4)$$

where $\varphi(k)$ is Euler's function, expressing the number of natural numbers $l \leq k$ with $(l, k) = 1$; here

$$S = \sum_{\chi} \bar{\chi}(l) \sum_{\rho_{\chi}} x^{-\delta} \exp\{[-\delta(3-\delta) - \tau^2 + i\tau(3-2\delta)]y + i\tau \log x\}$$

and $\rho_{\chi} = 1 - \delta + i\tau$ runs through all zeros of the functions $L(s, \chi)$ with $\delta \geq 0$. Putting in (4)

$$x = k^{\xi}, \quad \xi \geq 0, \quad y = \frac{\eta}{\nu} \log k \quad (5)$$

and dividing by $2\sqrt{\pi y} x^{3/2} e^{9y/4}$, we get

$$U = I_B \frac{\Phi(x, y; k, l)}{2\sqrt{\pi y} x^{3/2} e^{9y/4}} = 1 - I_{BS} + I_B \frac{O(k \log k)}{2\sqrt{\pi y} x^{3/2} e^{9y/4}}, \quad (6)$$

where I_B denotes integration with respect to η over the limits $\eta, \eta + 1$, repeated $B \geq \max(2, C + 1)$ times.

In (5) ν is restricted by the conditions $1 \leq \nu \leq \log k$, $\nu \leq e^{\alpha\xi}$, $\alpha \leq \min(1/10, A/3B)$.

In (6) I_{BS} is split into three parts corresponding to the conditions:

- 1) $|\tau| \geq \log k$;
- 2) $\delta \geq \lambda_0 / \log k$, $|\tau| \leq d(\delta) = \min(\log k, k^\delta / \log k)$;
- 3) $\delta \geq \lambda / \log k$, $d(\delta) < |\tau| < \log k$.

The term corresponding to $\rho = \beta_1$ is considered separately. By means of (2), using Abel's identity expressing a sum through an integral, it is proved that, if the initial value $\eta = \eta_0 \geq \max(4, Ce^{3\alpha C}, 1/A)$ is sufficiently large, then

$$U > \frac{c_2}{\nu} \delta_0 \log k. \quad (7)$$

Let $z = xe^{4y}$, and let V be the part of the sum U corresponding to the values $n = p \in (x, z)$. With the aid of Abel's identity it is proved that

$$U - V < c_3 e^{-(\eta_0/4\nu) \log k}. \quad (8)$$

Using Siegel's theorem and putting $c' = 1/\alpha$, $c = 4(\eta_0 + B)$, one can show that, under the conditions of our theorem, the right-hand side of (8) is smaller than the right-hand side of (7). Hence $V > 0$, from which the assertion immediately follows.

A simple consequence of the inequalities proved is the estimate

$$\pi(x; k, l) > x/\varphi(k)k^{3\varepsilon} \quad \text{for } k > k_1(\varepsilon), \quad x \in (k^{c' \log(c/\varepsilon)}, e^{k^\varepsilon}),$$

where $\pi(x; k, l)$ denotes the number of primes $p \leq x$, $p \equiv l \pmod{k}$ ($(k, l) = 1$).

Let $\lambda(n) = (-1)^v$, where v is the number of all prime factors of n , counting multiple factors separately. Further, let $\mu(n) = \lambda(n)$ if n is not divisible by a square > 1 , and let $\mu(n) = 0$ for all other n . An obvious consequence of the theorem is the existence of such a and b ($a \leq 2c'$, $b \leq 2c$) that, for any positive $\varepsilon \leq b$, for all $k > k_2(\varepsilon)$ and all $x \geq k^{a \log(b/\varepsilon)}$, the functions $\lambda(m)$ and $\mu(m)$ change sign in the interval (x, xk^ε) , when m runs over the numbers $\equiv l \pmod{k}$, and in the same interval there is a prime number $\equiv l \pmod{k}$ ($(k, l) = 1$).

The theorem is of interest for $x < \exp k^{\varepsilon_1}$ (ε_1 an arbitrarily small positive constant). For $x \geq \exp k^{\varepsilon_1}$, by the same means one can prove that at least one $p \equiv l \pmod{k}$ ($(a, l) = 1$) is found in the interval (x, xc_4) for some $c_4 > 1$.

Using stronger results on the distribution of zeros of Dirichlet L -functions, N. G. Chudakov proved ⁴ the existence of a prime number $p \equiv l \pmod{k}$ ($(k, l) = 1$) in the interval

$$(x, x(1 + x^{-\vartheta})), \quad \vartheta = \frac{1}{4} \quad (x \geq \exp k^{\varepsilon_1}), \quad (9)$$

but for $x < \exp k^{\varepsilon_1}$ his method is inapplicable. I proved ⁵ that for some $\vartheta > 1/3$, in the interval (9) the functions $\Lambda(m)$, $\lambda(m)$, $\mu(m)$ ($m \equiv l \pmod{k}$) preserve their mean values. A weak analogue of the latter for $x < \exp k^{\varepsilon_1}$ is provided by the results of the present note.

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CITED LITERATURE

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- ⁴ N. G. Chudakov, *Izv. AN SSSR, ser. matem.*, **12**, 31 (1948).
- ⁵ E. K. Fogels, *Izv. AN LatvSSR*, **4** (9), 109 (1948).

Note: Figure translations are in progress. See original paper for figures.

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