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Abstract

Full Text

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ON J-HERMITIAN OPERATORS

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1. Let H be a separable Hilbert space with scalar product (x, y) $(x, y \in H)$; let P and Q be orthogonal projections in H, with $P + Q = I$ (I is the identity mapping), and let $J = P - Q$. We define in H the indefinite scalar product $[x, y]$ by the formula (cf. ([1-4]))

$$[x, y] = (Jx, y) \quad (x, y \in H). \tag{1}$$

By an operator in H we mean a linear bounded mapping of H into H. An operator A in H will be called J-Hermitian ([4]) if $[Ax, y] = [x, Ay]$ for all $x, y \in H$. An operator A is J-Hermitian if and only if $A = JA^*J$, where A^* is the operator adjoint to A. Hence it follows that, for a J-Hermitian operator A,

$$\operatorname{Im} A = \frac{1}{2i}(A - A^*) = \frac{1}{i}(PAQ + QAP), \tag{2}$$

and, conversely, every operator satisfying condition (2) is J-Hermitian. An operator U in H is called J-unitary ([3,4]) if there exists an everywhere-defined inverse operator in H and $[Ux, Uy] = [x, y]$ for all $x, y \in H$.

The spectrum $\sigma(A)$ of an operator A and its parts $\sigma_p(A)$, $\sigma_c(A)$, $\sigma_r(A)$, as well as the resolvent set $\rho(A)$, are defined as in ([5]). Then it is not difficult to prove the following proposition:

Theorem 1. If the operator A in H is J-Hermitian, then:

1) the set $\sigma(A)$ is symmetric with respect to the real axis (cf. ([2])).

2) From $\lambda \in \sigma_p(A)$ it follows that $\overline{\lambda} \in \sigma_p(A) \cup \sigma_r(A)$.

3) From $\lambda \in \sigma_r(A)$ it follows that $\overline{\lambda} \in \sigma_p(A)$.

Corollary. For a J-Hermitian operator A, the set $\sigma_p(A) \cup \sigma_r(A)$ is symmetric with respect to the real axis, and $\sigma_r(A)$ contains no real points.

2. L. S. Pontryagin proved (see also ([2,6,7])) that for a J-Hermitian operator A, under the condition that $\dim PH = \varkappa$ $(0 < \varkappa < \infty)$, there always exists a nonnegative invariant $\varkappa$-dimensional subspace in which the spectrum of

A has a nonnegative imaginary part. We generalize this theorem to J-Hermitian operators with a completely continuous imaginary part (for $\varkappa = \infty$).

A. We denote

$$T^+ = \{x : x \in H,\ [x, x] > 0\} \cup \{0\};$$

$$T^- = \{x : x \in H,\ [x, x] < 0\} \cup \{0\}; \qquad T^0 = \{x : x \in H,\ [x, x] = 0\}.$$

If T is a subset of H, then a subspace contained in T will be called a **maximal subspace** of T if every other subspace containing the given one is not wholly contained in T.

We shall consider operators E in H with the properties:

$$EP = E, \qquad PE = P. \tag{3}$$

Obviously, $E^2 = E$. If the range of E is contained in $T^+ \cup T^0$ and conditions (3) are satisfied, then EH is a maximal subspace of $T^+ \cup T^0$. A simple metric characteristic of such operators E is given by:

Lemma 1. *If for an operator E in H conditions (3) are satisfied, then $EH \subset T^+ \cup T^0$ if and only if $\|E\| \le \sqrt{2}$.*

B. Here and everywhere below in Section 2 we assume $\dim PH = \dim QH = \infty$. In the range PH of the orthogonal projector P, introduce an orthonormal basis $x_1, x_2, \ldots$. By P_n denote the orthogonal projector from H onto the linear span of the vectors $x_1, x_2, \ldots, x_n$ $(n = 1, 2, \ldots)$. Let $I_n = P_n + Q,\ J_n = P_n - Q,\ H_n = I_n H$. The definite (respectively, indefinite) scalar product in H induces a definite (respectively, indefinite) scalar product in H_n, which we shall again denote by (x, y) (respectively, $[x, y]$) $(x, y \in H_n)$. The relation $[x, y] = (J_n x, y)$ holds. An operator U_n (respectively, A_n) in H_n is again called J_n-unitary (respectively, J_n-Hermitian) if U_n^{-1} exists as an operator in H_n, and $[U_n x, U_n y] = [x, y]$ (respectively, $[A_n x, y] = [x, A_n y]$) for all $x, y \in H_n$.

Lemma 2. *If U_n is a J_n-unitary operator in H_n, and for an operator E_n in H_n the conditions $E_n P_n = E_n,\ P_n E_n = P_n$ and $\|P_n E_n x\| - \|Q E_n x\| \ge 0$ for $x \in H_n$ are satisfied, then the operator $P_n U_n^{-1} E_n P_n$ maps the subspace $P_n H_n$ one-to-one onto itself.*

B. Denote by $\mathfrak{T}$ the totality of all operators E_n in H_n possessing the properties

$$\|P_n E_n x\| - \|Q E_n x\| \ge 0 \qquad (x \in H_n); \tag{4}$$

$$P_n E_n = P_n, \qquad E_n P_n = E_n. \tag{5}$$

The set $\mathfrak{T}$ is nonempty, since $P_n \in \mathfrak{T}$. It is easily proved that $\mathfrak{T}$ is convex and closed in the weak topology. Moreover, from Lemma 1, for $E_n \in \mathfrak{T}$ it follows that $\|E_n\| \leq \sqrt{2}$. But the unit sphere in the ring of operators in H^n is weakly compact ([5]). Therefore $\mathfrak{T}$, as a weakly closed part of a weakly compact set, is also compact in the weak operator topology.

G. If U_n is a J_n-unitary operator in H_n, then, by Lemma 2, there exists a bounded one-to-one mapping $(P_n U_n^{-1} E_n P_n)^{-1}$ of the subspace $P_n H_n$ onto $P_n H_n$.

Consider the mapping D defined on $\mathfrak{T}$:

$$D(E_n) = U_n^{-1} E_n P_n (P_n U_n^{-1} E_n P_n)^{-1} P_n, \qquad E_n \in \mathfrak{T}.$$

Then $P_n D(E_n) = P_n,\ D(E_n) P_n = D(E_n),\ \|P_n D(E_n) x\| - \|Q D(E_n) x\| \geq 0$ for $x \in H_n$, i.e. $D(E_n) \in \mathfrak{T}$. Moreover, the mapping D is continuous in the weak operator topology, since $P_n H_n$ is finite-dimensional. By the Schauder–Tikhonov theorem ([5]), the mapping D has at least one fixed point. Thus, there exists an operator $E_n^{(0)} \in \mathfrak{T}$ such that $D(E_n^{(0)}) = E_n^{(0)}$, and, consequently,

$$U_n E_n^{(0)} = E_n^{(0)} U_n E_n^{(0)}. \tag{6}$$

Thus, $E_n^{(0)} H_n$ is an n-dimensional nonnegative invariant subspace for U_n.

Using the Cayley transform ([2,8]), it follows from this that for every J_n-Hermitian operator A_n in H_n there exists an operator $E_n^{(0)}$ with the properties:

$$\|P_n E_n^{(0)} x\| - \|Q E_n^{(0)} x\| \geqslant 0 \quad (x \in H_n); \tag{4'}$$

$$E_n^{(0)} P_n = E_n^{(0)}, \qquad P_{nE} n^{(0)} = P_n; \tag{5'}$$

$$E_n^{(0)} A_{nE} n^{(0)} = A_{nE} n^{(0)}. \tag{6'}$$

Inequality $(4')$, by virtue of $(5')$ and Lemma 1, is equivalent to $\|E_n^{(0)}\| \leqslant \sqrt{2}$.

D. Let A be a J-Hermitian operator in H with a completely continuous imaginary part, and let I_n, J_n, H_n be defined as in B. Further, let $E_n = I_{nAI} n$. To the operator A_n there corresponds in H_n an operator $E_n^{(0)}$ with properties $(4')$, $(5')$, $(6')$. Define an operator E_n' in H by the equality $E_n' = E_n^{(0)} I_n$. Then for E_n' relations analogous to $(4')$ and $(5')$ are valid, and

$$E_n' A E_n' x = I_{nAE} n' x \quad (x \in H). \tag{7}$$

From $\|E'_n\| \leqslant \sqrt{2}$ there follows the existence of a weakly convergent subsequence (E'_{n_ν}), i.e. there exists an operator E_0 in H such that $E'_{n_\nu} \to E_0$ (as $\nu \to \infty$) in the weak operator topology. Again $\|E_0\| \leqslant \sqrt{2}$ and $E_0 P = E_0,\ PE_0 = P$. From (7), taking into account relation (2), and also the fact that $\operatorname{Im} A$ is completely continuous, passing to the limit as $\nu \to \infty$, we obtain $E_0 A E_0 = AE_0$. Hence, finally, we obtain:

Theorem 2. Let A be a J-Hermitian operator in H with a completely continuous imaginary part. Then there exists an (idempotent) operator E_0 in H with the properties

$$\|E_0\| \leqslant \sqrt{2}, \qquad PE_0 = P, \qquad E_0 P = E_0, \qquad AE_0 = E_0 A E_0.$$

Consequently, there exists a maximal subspace $E_0 H$ of $T^+ \cup T^0$, invariant with respect to A.

Corollary. If A satisfies the conditions of Theorem 2 and $E_0^+ = JE_0^* J$, then the subspace $(I - E_0^+)H$ is also invariant with respect to A. $(I - E_0^+)H$ is a maximal subspace of $T^- \cup T^0$.

Starting from the subspace $E_0 H$, one can construct an infinite-dimensional subspace M, invariant with respect to A, such that the spectrum of A in M has a nonnegative imaginary part.

3. In this section we consider the case when $\varkappa = 1$.

The operator A is called **simple** ([9]) if there exists a vector x_1 such that c.l.s. $[A^p x_1,\ p = 0, 1, ...] = H$, i.e. if A is a **cyclic operator** ([10]) with generating element x_1. Every J-Hermitian operator in H in the case $\varkappa = 1$ is representable as the orthogonal sum of a simple J-Hermitian operator with generating vector $x_1 \in PH$ and an (ordinary) Hermitian operator. Therefore we shall assume that A is a simple operator, and denote $\lambda_1 = (Ax_1, x_1)$.

Using the model constructed in ([9]), pp. 24-27, we obtain:

Theorem 3. Let A be a simple J-Hermitian operator in the space H ($J = P - Q,\ \dim PH = 1$). Then A is unitarily equivalent to the operator

$$\widetilde{A}\{\gamma, g(t)\} = \left\{\lambda_1 \gamma - \int_{-\infty}^{\infty} g(t)\, d\tau(t),\ tg(t) + \gamma\right\}$$

in the space $\widetilde{H} = R_1 \times L^2_\tau$ with scalar product

$$(\{\varphi, f\}, \{\gamma, g\}) = 2\left[\varphi\overline{\gamma} + \int f(t)\overline{g(t)}\, d\tau(t)\right].$$

To the element x_1 there corresponds the element

$$\left\{\frac{1}{\sqrt{2}}, 0\right\} \in \widetilde{H}.$$

$\tau(t)$ is defined as

$$\tau(t) = (E_t QAx_1,\ QAx_1), \tag{8}$$

where E_t is the resolution of the identity for $\dfrac{A+A^*}{2}$.

With the aid of the representation of a simple J-Hermitian operator A established in Theorem 3, for the case $\dim PH = 1$ one can prove certain assertions concerning the spectrum of A. For the eigenvalues of the operator the following holds:

A number λ is an eigenvalue of A if and only if it satisfies the equation

$$\int \frac{d\tau(t)}{t-\lambda} + \lambda_1 - \lambda = 0$$

and, moreover,

$$\int \frac{d\tau(t)}{|t-\lambda|^2} < \infty.$$

The limiting points of $\sigma(A)$ can only be points of $\sigma\left(\dfrac{A+A^*}{2}\right)$, since the imaginary part of A is finite-dimensional $(*)$. In view of (8), the relation

$$\sigma\left(\frac{A+A^*}{2}\right) = S_\tau \cup \{\lambda_1\}$$

holds, where S_τ is the set of growth points of the function τ. From (9) it follows that the limiting points S_τ coincide with the limiting points of $\sigma\left(\dfrac{A+A^*}{2}\right)$.

Theorem 4. Under the assumptions of Theorem 3, every limiting point S_τ that is not an eigenvalue of A belongs to $\sigma_c(A)$.

Examples show that isolated growth points $\tau(t)$ which are isolated eigenvalues of $\sigma\left(\dfrac{A+A^*}{2}\right)$ may belong to $\rho(A)$.

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Note: Figure translations are in progress. See original paper for figures.