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Abstract

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PROPERTIES OF THE CLASSES W_p^r WITH FRACTIONAL DERIVATIVE ON DIFFERENTIABLE MANIFOLDS

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This work is a further development of the results published in our paper (^{1}).

Let g be a domain in the space R_n; R_m is the m-dimensional subspace of points $(x_1, \dots, x_m, x_{m+1}, \dots, x_n)$, where $x_{m+1}, \dots, x_n$ are fixed.

A function f belongs to the class $W_p^r(g)$, $1 < p < \infty$, for $r = \bar{r} + \alpha$, where $\bar{r}$ is an integer, $0 < \alpha < 1$, if:

1) f belongs to the Sobolev class $W_p^{\bar{r}}(g)$;

2)
$$\|f^{\bar{r}}\|_{W_p^\alpha(g)} = \iint\limits_{g\,g} \frac{|f^{\bar{r}}(x_1, \dots, x_n) - f^{\bar{r}}(y_1, \dots, y_n)|^p}{|x-y|^{\,n+p\alpha}}\, dg_x\, dg_y < \infty,$$

where
$$|x-y| = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2}, \quad i = 1, 2, \dots, n.$$

Put
$$\|f\|_{W_p^r(g)} = \|f\|_{W_p^{\bar{r}}(g)} + \sum \|f^{\bar{r}}\|_{W_p^\alpha(g)}.$$

The following theorems for the classes $W_p^r(R_n)$ are analogous to the corresponding theorems for the classes H_p^r of S. M. Nikol' skii (^{2}).

Theorem 1 (direct). Let $f \in W_p^r(R_n)$, $1 \le m < n$, $\lambda_j \ge 0$,

$$\rho = r - \frac{(n-m)}{p} - \sum_{j=m+1}^{n} \lambda_j > 0.$$

Then the partial derivative

$$\psi(x_1, \dots, x_m) = \frac{\partial^{\lambda_{m+1} + \dots + \lambda_n} f(x_1, \dots, x_n)}{\partial x_{m+1}^{\lambda_{m+1}} \cdots \partial x_n^{\lambda_n}}$$

as a function of $x_1, \dots, x_m$, for any fixed $x_{m+1}, \dots, x_n$, belongs to $W_p^\rho(R_m)$ (ρ not an integer when $p > 2$),

$$\|f\|_{W_p^\rho(R_m)} \le c\|f\|_{W_p^r(R_n)},$$

where c does not depend on f.

Theorem 2 (converse). Given a number $r > 0$ and all possible systems $\{\lambda\}$ of nonnegative numbers $\lambda_{m+1}, \dots, \lambda_n$, for which

$$\rho^\lambda = r - \frac{(n-m)}{p} - \sum_{j=m+1}^{n} \lambda_j \ge 0.$$

Let to each system $\{\lambda\}$ there correspond a function $\varphi_\lambda(x_1, \dots, x_m)$ belonging to $W_p^\rho(R_m)$ (ρ nonintegral for $1 < p < 2$). Then one can construct a function $f(x_1, \dots, x_n) \in W_p^r(R_n)$ such that

$$\left.\frac{\partial^{\lambda_{m+1}+\dots+\lambda_n} f(x_1, \dots, x_n)}{\partial x_{m+1}^{\lambda_{m+1}} \cdots \partial x_n^{\lambda_n}}\right|_{R_m} = \varphi_\lambda(x_1, \dots, x_m),$$

$$\|f\|_{W_p^r(R_n)} \le c \sum_{\{\lambda\}} \|\varphi_\lambda\|_{W_p^\rho(R_m)}.$$

Theorems 1 and 2 for $p = 2$ are due to L. N. Slobodetskii (^{7}). S. M. Nikolskii in (^{3}) and V. M. Babich obtained a theorem on the extension of classes W_p^r (r integral) from a bounded domain g to the whole space R_n. The following theorem generalizes this result to the case where r is fractional (for $p = 2$ it is due to L. N. Slobodetskii).

Theorem 3. If $f \in W_p^r(G)$, G is bounded, then, whatever $\eta > 0$, there exists a function $F \in W_p^r(R_n)$ such that

$$F|_{G_\eta} = f, \qquad \|F\|_{W_p^r(R_n)} \le c\|f\|_{W_p^r(G)},$$

where c depends only on η.

If the boundary Λ of the domain G is sufficiently smooth ($\Lambda \in C^{(r+1)}$), then η may be set equal to 0.

With the aid of this theorem, by known methods proposed by S. M. Nikolskii (^{2}), Theorems 1 and 2, as well as all the theorems of our work (^{1}), are carried over to bounded domains G with sufficiently smooth boundary.

Definition. Let G be an n-dimensional bounded domain, and let Λ_m be its sufficiently smooth m-dimensional boundary. We shall say that a function $\sigma = \sigma(x_1, \dots, x_n)$, defined on G, satisfies the inequality

$$c_1\rho(x,\Lambda_m) \leq \sigma(x) \leq c_2\rho(x,\Lambda_m),$$

where $\rho(x,\Lambda_m)$ is the distance from the point x to Λ_m, and c_1, c_2 are constants depending only on the domain G. We shall say that a function $f = f(x_1, \dots, x_n)$ belongs to the class $W^r_{p,\alpha}(G,\Lambda_m)$, if it is defined on G, has on G all generalized partial derivatives up to order r (r integral) inclusive,

$$\|f\|_{W^r_{p,\alpha}(G,\Lambda_m)} = \sum_{l=0}^{r} \int_G \sigma^\alpha \sum_{\alpha_1+\dots+\alpha_n=l} \left| \frac{\partial^l f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right|^p dg < \infty, \qquad -k < \alpha < pr.$$

Such classes were considered in works $({}^{4-6})$, etc.

L. D. Kudryavtsev obtained extension and embedding theorems for the classes $W^r_{p,\alpha}(G,\Lambda_m)$ into $H^r_p(G)$. Below are formulated theorems on the extension and embedding of the classes $W^r_{p,\alpha}(G,\Lambda_m)$ into $W^r_p(G)$, which generalize the results of A. A. Vasharin $({}^5)$ and P. I. Lizorkin $({}^6)$.

Theorem 4. Let $f \in W^r_p(\Lambda_m)$ (r nonintegral for $1 < p < 2$), $\Lambda_m \in C^{l+1}$, $l > r$. Then there exists a function F, defined on G, such that

$$F\big|_{\Lambda_m} = f, \qquad F \in W^{\,l}_{p,\,p(l-r)-k}(G,\Lambda_m),$$

$$\|F\|_{W^{\,l}_{p,\,p(l-r)-k}(G,\Lambda_m)} \leq c\|f\|_{W^r_p(\Lambda_m)}.$$

Theorem 5. Let $f \in W^r_{p,\alpha}(G,\Lambda_m)$, and let Λ_k be a k-dimensional, sufficiently smooth closed manifold belonging to $\overline{G}$. Then, if

$$\rho = r - \frac{(n-k)}{p} - \sum_{i=1}^{n} \lambda_i - \frac{\alpha}{p} > 0, \qquad 1 \leq k < n,$$

then

$$\left.\frac{\partial^{\lambda_1+\dots+\lambda_n} f(x_1,\dots,x_n)}{\partial x_1^{\lambda_1} \cdots \partial x_n^{\lambda_n}}\right|_{\Lambda_k} = \psi(\Lambda_k) \in W^\rho_p(\Lambda_k)$$

(ρ nonintegral for $p > 2$),

$$\|\psi\|_{W^\rho_p(\Lambda_k)} \leq c\|f\|_{W^r_{p,\alpha}(G,\Lambda_m)}.$$

Theorems 4 and 5 were obtained by A. A. Vasharin for $p = 2$ and $m = k = n - 1$, and by P. I. Lizorkin for $0 \leq r < 1$, $m = k = n - 1$, $1 < p < \infty$.

Theorem 6. Let $f \in W^r_{p,\alpha}(G, \Lambda_m)$, $\Lambda_m \in C^{r+1}$. Then $f \in W_p^{r-\alpha/p}(G)$ $(\alpha \geq 0)$ and $f \in W^r_p(G)$ $(\alpha < 0)$,

$$\|f\|_{W_p^{r-\alpha/p}(G)} \leq c\|f\|_{W^r_{p,\alpha}(G,\Lambda_m)}.$$

Theorem 7. Let $f \in W^r_{p,\alpha}(G, \Lambda_m)$, $\Lambda_m \in C^{r+1}$. Then, if

$$\alpha - vp > -k, \qquad v = 1, 2, \dots, r,$$

then

$$f \in W^{r-v}_{p,\alpha-vp}(G, \Lambda_m),$$

$$\|f\|_{W^{r-v}_{p,\alpha-vp}(G,\Lambda_m)} \leq c\|f\|_{W^r_{p,\alpha}(G,\Lambda_m)},$$

$$\int_{\Lambda_m} \sum_{\alpha_1+\dots+\alpha_n=r-v} \left| \frac{\partial^{r-v} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right|^p d\Lambda_m = o\big(\rho^{-(\alpha-vp+k)}\big)\,.$$

This theorem strengthens the corresponding theorem of L. D. Kudryavtsev in work (4)*.

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* This work arose as a result of participation in the seminar of V. I. Kondrashov, L. D. Kudryavtsev, and S. M. Nikol'skii. In preparing the work for publication it became known that, independently and simultaneously, Theorem 6, Theorem 5 for ρ nonintegral and $m = k$, and also Theorem 2 for ρ nonintegral and Theorem 4 for r nonintegral had been obtained by another participant of the seminar, P. I. Lizorkin.

Note: Figure translations are in progress. See original paper for figures.