B. Ya. LIPKO

An extensive literature is devoted to the study of the solvability of mixed problems for the parabolic equation

1960

SovietRxiv

Abstract

Full Text

MATHEMATICS

B. Ya. LIPKO

ON A MIXED PROBLEM WITH OBLIQUE DERIVATIVE FOR A SECOND-ORDER PARABOLIC EQUATION

(Presented by Academician I. N. Vekua on 13 I 1960)

An extensive literature is devoted to the study of the solvability of mixed problems for the parabolic equation

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} A_{ij}(x,t)\frac{\partial^2 u}{\partial x_j \partial x_i} + \sum_{j=1}^{n} B_j(x,t)\frac{\partial u}{\partial x_j} + C(x,t)u \equiv Lu \tag{1}$$

In the main, problems have been subjected to detailed and comprehensive study in which, on the boundary of the domain, either the unknown function or a linear combination of the function and the derivative in the direction of the conormal is prescribed, while the domain has been assumed bounded. In the work of L. S. Slobodetskii ([2]), with the aid of generalized heat potentials, the correct solvability of the indicated exterior problems was established in classes of rapidly increasing functions for equations with bounded coefficients. In a recent work of M. Pani ([3]), in the case of the heat equation with two spatial variables, a special fundamental solution is constructed and, with its help, the correct solvability of the mixed problem with oblique derivative is proved for a finite domain bounded by a sufficiently smooth surface; here the domain may be noncylindrical in the space x_1, x_2, t.

In the present note we set forth theorems establishing the correct solvability of the mixed problem with oblique derivative for equation (1) both for finite and for infinite domains; moreover, in the case of the latter, the investigation also covers equations with coefficients increasing as the spatial coordinates increase. The proofs of these theorems are carried out with the aid of generalized heat potentials; in doing so, essential use is made of the construction of the fundamental solution for the case of constant coefficients, analogous to the above-mentioned construction of M. Pani, and of the methods developed in the works of S. D. Eidelman in the investigation of the Cauchy problem for systems parabolic in the sense of I. G. Petrovskii.

1. We shall consider the cylindrical domain $V = [0, T] \times D$ with lateral surface S, which may be divided into a finite number of pieces, the equation of

each of which is representable in a local coordinate system in the form $\xi_n = \psi(\xi_1, \dots, \xi_{n-1})$, with ψ continuous and bounded together with its second derivatives; D is a finite or infinite domain of the space $x_1, x_2, \dots, x_n$. To equation (1) we adjoin the following conditions:

$$u\Big|_{t=0} = 0, \qquad \alpha\frac{\partial u}{\partial l} + \beta u\Big|_S = f(x,t), \qquad (x,t) \in S, \quad t > 0, \tag{2}$$

where l is a direction prescribed at each point of the boundary surface S by the direction cosines $\alpha_1, \alpha_2, \dots, \alpha_n$, and such that the scalar product-

condition $(\mathbf{l}, \mathbf{n}) > 0$, $\mathbf{n}$ is the inner normal; α, β are continuous and bounded functions on S.

First consider the equation with constant coefficients depending on the parameters y and θ

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} A_{ij}(y,\theta)\frac{\partial^2 u}{\partial x_i \partial x_j}, \tag{3}$$

and the special fundamental solution of this equation

$$H_0(t,\tau,x;\xi,y,\theta) = \frac{(\vec{\lambda},\vec{\mu})}{(t-\tau)^{n/2}}\exp\left[-\frac{p^2}{4(t-\tau)}\right] - \frac{(\mathbf{m},\vec{\mu})(\mathbf{m},\mathbf{p})}{(t-\tau)^{(n+1)/2}}\exp\left[-\frac{p^2}{4(t-\tau)}\right]\left\{\frac{\Phi\left(\frac{(\vec{\lambda},\mathbf{p})}{2(t-\tau)^{1/2}}\right) - \Phi\left(\frac{(\vec{\lambda},\mathbf{p})+d}{2(t-\tau)^{1/2}}\right)}{\exp\left[-\frac{(\vec{\lambda},\mathbf{p})^2}{4(t-\tau)}\right]}\right\}; \tag{4}$$

$\vec{\lambda}$ is the vector with coordinates $\alpha'_k = \sum_{j=1}^n b_{kj}\alpha_j$; $\vec{\mu}$ is the vector with coordinates $\beta'_k = \sum_{j=1}^n a_{kj}\beta_j$; b_{kj} are the elements of the matrix $(A^{-1})^{1/2}$; a_{kj} are the elements of the matrix $A^{1/2}$; A is the matrix of coefficients of equation (1); β_j are the direction cosines of the inner normal; $\mathbf{p}$ is the vector with coordinates $\sum_{j=1}^n b_{kj}(x_j - \xi_j)$; $(x_1, \dots, x_n, t)$ are the coordinates of the point M inside the domain; $\xi_1, \xi_2, \dots, \xi_n, \tau$ are the coordinates of the point N on the surface S; the vector $\mathbf{m}$ lies in the two-dimensional plane formed by the vectors $\vec{\lambda}$ and $\vec{\mu}$; $(\vec{\lambda}, \mathbf{m}) = 0$; $\Phi(z) = \int_0^z e^{-y^2}\,dy$; $2d$ is the lower bound of the distances from the point N to the point at which the axis opposite to $\mathbf{l}$ meets the surface S. The function $H_0(t,\tau,x,\xi,y,\theta)$ has the following properties: 1) $H_0(t,\tau,x,\xi,y,\theta)$ is continuous together with its derivatives with respect to $x_1, \dots, x_n$, if the point M lies inside the domain; 2) $H_0(t,\tau,x,\xi,y,\theta)$, as a function of the arguments

t and x, is a solution of equation (3); 3) $\frac{\partial H_0}{\partial l} = \frac{\partial G_0}{\partial \nu} + L(t, \tau, x, \xi, y, \theta)$, where $G_0 = \frac{1}{(t-\tau)^{n/2}} \times$

$$\times \exp\left[-\frac{p^2}{4(t-\tau)}\right];$$

ν is the conormal direction; $L(t, \tau, x, \xi, y, \theta)$ is a function continuous in V up to the boundary; 4) the estimate holds

$$\left|\frac{\partial^m H_0}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}}\right| \leq C_0 (t-\tau)^{-\frac{n+m}{2}} \exp\{-c_0 |x-\xi|^2 (t-\tau)^{-1}\}, \quad m = 0, 1, 2, \ldots \tag{5}$$

We seek the function $H(t, \tau, x, \xi)$ for problem (1), (2) in the form

$$H(t, \tau, x, \xi) = H_0(t, \tau, x, \xi, \xi, \tau) + \int_\tau^t d\beta \int_D G_0(t-\beta, x-y, y, \beta) \varphi(\beta, \tau, y, \xi)\, dy; \tag{6}$$

then, with respect to $\varphi(t, \tau, x, \xi)$, we obtain the integral equation

$$\varphi(t, \tau, x, \xi) = F(t, \tau, x, \xi) + \int_\tau^t d\beta \int_D K(t, \beta, x, y) \varphi(\beta, \tau, y, \xi)\, dy, \tag{7}$$

$$F(t, \tau, x, \xi) = \left(\frac{\partial}{\partial t} - L\right) H_0, \qquad K(t, \tau, x, \xi) = \left(\frac{\partial}{\partial t} - L\right) G_0.$$

Under certain conditions on the coefficients of equation (1), formulated below, F and K have estimates which make it possible, by means of the method of (1), to obtain for φ the estimate

$$|\varphi| \leq C(t-\tau)^{-\frac{n+1+\alpha}{2}} \exp\left\{-c\,\frac{|x-\xi|^2}{t-\tau}\right\}.$$

It follows from this estimate that the function $H(t, \tau, x, \xi)$ possesses properties 1), 2), 3), 4) of the function H_0. This makes it possible to reduce problem (1), (2), by means of the potential

$$\int_0^t d\tau \int_S H(t, \tau, x, \xi) \mu(\xi, \tau)\, dS$$

to an integral equation, for which the usual method of successive approximations applies.

Theorem 1. *Suppose: 1) the coefficients of equation (1) are given in the domain V, are continuous and bounded in it, and the continuity of the coefficients with respect to x and t is understood in the Hölder sense; 2) $f(x,t)$ is continuous on S and $|f(x,t)| \le M_0 e^{k|x|^2}$.*

Then there exists a solution $u(x,t)$ of problem (1), (2) such that

$$|u(x,t)| \le M e^{k_1 |x|^2}.$$

2. In this paragraph the question of uniqueness of the solution of problem (1), (2) will be considered in the case of an unbounded domain V. Using a formula of the type of formula (7), (5) from (5), and a method analogous to that developed in (4) in the case of the Cauchy problem, we obtain the following theorem:

Theorem 2. *Suppose the following conditions are satisfied: 1) the coefficients $A_{ij}(x,t)$ are given in the domain V, are twice differentiable with respect to $x_1, \dots, x_n$, and their second derivatives are Hölder continuous with respect to x and t; 2) the $B_j(x,t)$ are given in V and have there Hölder-continuous first derivatives with respect to $x_1, \dots, x_n$, bounded in V; 3) $C(x,t)$ is given in V, bounded, and continuous in the Hölder sense.*

Then every solution $u(x,t)$ of equation (1) such that

$$|u(x,t)| < e^{c\,r\,h(r)}, \qquad u\big|_{t=0} = 0, \qquad \alpha \frac{\partial u}{\partial l} + \beta u\big|_S = 0,$$

where

$$r = \left(\sum_{s=1}^{n} x_s^2 \right)^{1/2}, \qquad \int_1^{\infty} \frac{dr}{h(r)} = \infty,$$

is identically equal to zero.

3. Let us consider problem (1), (2) in the case where the coefficients of equation (1) grow as the spatial coordinates increase. Suppose: 1) $A_{ij}(x,t)$ and $C(x,t)$ are given in V and are twice differentiable with respect to $x_1, \dots, x_n$, their second derivatives being continuous in the Hölder sense, while $B_j(x,t)$ are given in V and continuous in the Hölder sense; 2)

$$|A_{ij}(x,t)| < A_0, \qquad |B_j(x,t)| < B_0 f^{1-\varepsilon}(x),$$

$$C(x,t) < -C_0 f^2(x);$$

A_0, B_0, C_0 are positive constants; $0 < \varepsilon < 1$; $f(x)$ is a positive continuous function. Under these assumptions let us consider the equation

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} A_{ij}(y,\theta)\frac{\partial^2 u}{\partial x_i \partial x_j} + C(y,\theta)u \tag{8}$$

and the function

$$H_1(t,\tau,x,\xi,y,\theta) = H_0(t,\tau,x,\xi,y,\theta)e^{C(y,\theta)(t-\tau)},$$

where $H_0(t,\tau,x,\xi,y,\theta)$ is from (4). For it the estimate holds

$$|H_1| \le C_1(t-\tau)^{-\frac{n}{2}} \exp\left\{-\frac{c_1|x-\xi|^2}{t-\tau} - c_0 f^2(y)(t-\tau)\right\}.$$

We shall seek the function $H(t,\tau,x,\xi)$ in the form

$$H(t,\tau,x,\xi) = H_1(t,\tau,x,\xi,\tau,x) + \int_\tau^t d\beta \int_D G_0(t,\beta,x,y)\varphi(\beta,\tau,y,\xi)\,dy;$$

then, with respect to $\varphi(t,\tau,x,\xi)$, we obtain the integral equation (7), and moreover

$$\max\{|F|,|K|\} \le C_2(t-\tau)^{-\frac{n+1+\alpha}{2}} \exp\left\{-\frac{c|x-\xi|^2}{t-\tau}\right\},$$

whence it follows that the method of (1) applies to equation (7). With the aid of the potential

$$\int_0^t d\tau \int_S H(t,\tau,x,\xi)\mu(\xi,\tau)\,dS$$

the problem is reduced to an integral equation, for which the usual method of successive approximations applies.

Remark. The arguments carried out in this section, in combination with the method developed in the work of S. D. Eidelman (1), make it possible to establish the solvability of the mixed problem with an oblique derivative in the case when the coefficients $B_j(x,t)$ and $C(x,t)$ grow respectively as $|x|$ and $|x|^2$.

In conclusion, I express my gratitude to my teacher S. D. Eidelman for posing the problem and for his attention during its solution.

Received
6 I 1960

CITED LITERATURE

[1] S. D. Eidelman, DAN, 120, No. 5 (1958); 127, No. 4 (1959). [2] L. N. Slobodetskii, DAN, 103, No. 1 (1955). [3] Mauro Pagni, Ann. Scuola Norm. Super. Pisa, 11, No. 1–2 (1957). [4] G. N. Zolotarev, Izv. vyssh. uchebn. zaved., Matematika, No. 2 (3) (1958). [5] K. Miranda, *Equations with Partial Derivatives of Elliptic Type*, IL, 1957.

Note: Figure translations are in progress. See original paper for figures.