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Abstract

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LIMITING SOLUTIONS OF A NONLINEAR EQUATION OF PARABOLIC TYPE

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1. We consider a nonlinear equation of parabolic type in the form (1)

$$\frac{1}{a}\frac{\partial F}{\partial t} = F\Delta F + \frac{1}{k}(\nabla F)^2. \tag{1}$$

For the one-dimensional case equation (1) takes the form

$$\frac{1}{a}\frac{\partial F}{\partial t} = F\left(\frac{\partial^2 F}{\partial r^2} + \frac{\nu - 1}{r}\frac{\partial F}{\partial r}\right) + \frac{1}{k}\left(\frac{\partial F}{\partial r}\right)^2, \tag{2}$$

where, as usual, the parameter ν assumes the values $1, 2, 3$, respectively, for plane, cylindrical, and spherical solutions. It is possible, however, to assign meaning to any other values of ν, including negative ones, if in the quasi-one-dimensional approximation one considers motion in a tube of variable cross section $S(r) \sim r^{\nu-1}$.

2. Let us consider a solution of the type of a wave converging to the singular point $r = 0$ of equation (1). We take the focusing time to be zero, so that the quantity $\tau = -t$ varies in the range $0 \leq \tau < \infty$. Instead of (2) we obtain the equation

$$\frac{1}{a}\frac{\partial F}{\partial \tau} + F\left(\frac{\partial^2 F}{\partial r^2} + \frac{\nu - 1}{r}\frac{\partial F}{\partial r}\right) + \frac{1}{k}\left(\frac{\partial F}{\partial r}\right)^2 = 0. \tag{3}$$

We seek the solution in self-similar form

$$F(r, \tau) = \frac{r^2}{a\tau}\varphi(\xi), \tag{4}$$

where $\xi = r/A\tau^\alpha$, A is a constant, and α is the exponent to be determined.

For the function $\varphi(\xi)$, according to (3), we obtain the equation in total derivatives

$$\varphi\left[\xi^2\frac{d^2\varphi}{d\xi^2}+(\nu+3)\xi\frac{d\varphi}{d\xi}+2\nu\varphi\right]+\frac{1}{k}\left(\xi\frac{d\varphi}{d\xi}+2\varphi\right)^2-\left(\alpha\xi\frac{d\varphi}{d\xi}+\varphi\right)=0, \quad (5)$$

which, by the substitution $p=\xi\, d\varphi/d\xi$, is reduced to the first-order equation

$$\frac{dp}{d\varphi}=-\frac{1}{p\varphi}\left[2\left(\nu+\frac{2}{k}\right)\varphi^2+\left(\nu+2+\frac{4}{k}\right)p\varphi+\frac{1}{k}p^2-\alpha p-\varphi\right]. \quad (6)$$

The independent variable is recovered by the quadrature

$$\ln\xi=\int_0^{\varphi}\frac{d\varphi}{p(\varphi)}. \quad (7)$$

The lower limit in this integral is chosen on the basis that at the wave front there must be $\varphi=0$, and by changing the scale of the coordinates and time one can arrange that the wave front correspond to $\xi=1$. In this case the inequality $1\le\xi\le\infty$ is satisfied.

The condition that the flux $q=\mathrm{const}\cdot F^{1/k}\dfrac{\partial F}{\partial r}$ vanish at the wave front gives

$$\varphi^{1/k}\xi\frac{d\varphi}{d\xi}=0 \quad \text{for } \xi=1. \quad (8)$$

We shall seek the expansion of $\varphi(x)$ as $x=\ln\xi\to 0$ in the form $\varphi(x)=x^{\lambda}(C_0+C_1x+\cdots)$, where $\lambda\neq 0$ and $C_0\neq 0$. Substituting this expansion into (5), and in the quadratic and linear groups of terms retaining the terms containing x to the lowest power, we then obtain

$$\lambda C_0\left\{\left(\lambda\frac{k+1}{k}-1\right)C_0x^{\lambda-1}-\alpha\right\}=0. \quad (9)$$

Moreover, from (8) it follows that

$$\lim_{x\to 0}\lambda C_0^{1+\frac{1}{k}}x^{\lambda(1+\frac{1}{k})-1}=0,$$

i.e. $\lambda>\dfrac{k}{k+1}$.

The two possibilities, $\lambda>1$ and $\lambda<1$, according to (9), lead respectively to the conditions $\lambda C_0^2=0$ and $\lambda C_0\alpha=0$, which contradict the initial assumptions. Hence it follows that $\lambda=1$ and $C_0=\alpha k$. Instead of (8) we now have

$$p=\xi\frac{d\varphi}{d\xi}=\alpha k \quad \text{for } \xi=1. \quad (10)$$

Fig. 1

Figure 1: Fig. 1

We seek a solution which at the moment $\tau = 0$ would give a finite value for $F(r, 0)$ when $r \neq \infty$. For $r \neq 0$ and $\tau \to 0$, the quantity $\xi = r/A\tau^\alpha \to \infty$. From (4) we conclude that

$$\varphi(\xi) = \text{const} \cdot \xi^{-1/\alpha} \quad \text{as } \xi \to \infty. \tag{11}$$

Then $F(r, 0) = \text{const} \cdot r^{2-1/\alpha}$, and since $F(0, 0) = 0$, it follows that $\alpha > 1/2$. From physical considerations it is clear that the required solutions correspond to accelerated motion of the wave front, which gives $\alpha < 1$.

From condition (11) it follows that as $\xi \to \infty$, $\varphi \to 0$ and $p = \xi\, d\varphi/d\xi = -\varphi/\alpha$. Thus, the sought integral curve of equation (6) must leave the point ($\varphi = 0;\ p = \alpha k$) and enter the origin ($\varphi = 0;\ p = 0$) along the straight line $\alpha p + \varphi = 0$.

Fig. 1

The character of the integral curves of equation (6) is shown in Fig. 1. This equation has three singular points: $L(\varphi = 0,\ p = 0)$, $M(\varphi = 0;\ p = \alpha k)$, $N\left(\varphi = \dfrac{1}{2\nu + 4/k};\ p = 0\right)$. The coordinate axes (except for the singular points) are isoclines of ∞. The ordinate axis, although it is an integral curve, is of no interest.

The singular point M is of saddle type; its second separatrix (the first is the line $\varphi = 0$) near M has the form:

$$p = \alpha k + A_1\varphi + A_2\varphi^2 + A_3\varphi^3 + \cdots, \tag{12}$$

where

$$A_1 = -\frac{1}{\alpha(k+1)}\left[\alpha k\left(\nu + 2 + \frac{4}{k}\right) - 1\right],$$

$$A_2 = -\frac{1}{\alpha(2k+1)}\left[2\left(\nu + \frac{2}{k}\right) + \frac{A_1}{\alpha k}\right],$$

$$A_3 = -\frac{A_2}{\alpha(3k+1)}\left[\frac{1}{\alpha k} + \left(2 + \frac{1}{k}\right)A_1\right].$$

The character of the singular point N depends on α. For $1/2 < \alpha < 1$ it is a focus. The origin of coordinates (the point L) is a complicated singular point

(?). Along the direction $\alpha p + \varphi = 0$, for $\varphi > 0$, only one integral curve (a separatrix) enters the point L, for which we have the expansion

$$p = B_1\varphi + B_2\varphi^2 + B_3\varphi^3 + \dots, \tag{13}$$

where

$$B_1 = -\frac{1}{\alpha}, \qquad B_2 = \frac{2}{\alpha^3}\left(\nu + \frac{2}{k}\right)\left(\alpha - \frac{1}{2}\right)\left(\alpha - \frac{k+1}{k\nu+2}\right) > 0,$$

$$B_3 = \frac{B_2}{\alpha}\left[\nu + 2 + \frac{4}{k} - \frac{1}{\alpha}\left(3 + \frac{2}{k}\right)\right].$$

For $\varphi < 0$, a multitude of integral curves enters along the direction $\alpha p + \varphi = 0$. Along the direction $\varphi = 0$, one integral curve enters the point L. Since $F(r,\tau)$ is an essentially positive quantity, the behavior of the integral curves in the half-plane $\varphi > 0$ is of practical interest.

The method of numerical computation consisted in starting from the points M and L with the aid of the expansions (12) and (13) and, by selecting the value of α, achieving the joining of the two branches of the integral curve on the ray $p = 0,\ \varphi > 0$. It turned out that small changes in the exponent α lead to a considerable divergence of the two branches. Therefore, with a small number of trials, for given ν and k one can obtain the exponent α with sufficient accuracy. Table 1 gives the values of α for $\nu = 2, 3$ and $k = 1, 2, \dots, 10$.

Table 1

ν	$k =$ 1	2	3	4	5	6	7	8	9	10
2	0.8567	0.8017	0.7624	0.7396	0.7231	0.7114	0.7003	0.6920	0.6818	0.6793
3	0.7682	0.7101	0.6598	0.6378	0.6217	0.6098	0.6005	0.5938	0.5869	0.5821

In Figs. 2 and 3, respectively for $\nu = 2$ and $\nu = 3$ and the same values of k, graphs are given of the function $\varphi(\eta)$, where $\eta = \xi^{-1/\alpha}$, constructed according to formula (7).

3. The self-similar solutions constructed are limiting in the sense that, if at some moment $\tau > 0$ there is a region with center at the point $r = 0$ such that throughout the region $F(r,\tau) = 0$, then, provided that dependence on other spatial coordinates is absent, the solution in a neighborhood of the point $r = 0,\ \tau = 0$ will necessarily enter a self-similar regime with the corresponding exponent α.

Fig. 2 and Fig. 3: plots of φ versus η for $\nu = 2$ and $\nu = 3$, with curves labeled $k = 1, \dots, 10$.

Figure 2: Fig. 2 and Fig. 3: plots of φ versus η for $\nu = 2$ and $\nu = 3$, with curves labeled $k = 1, \dots, 10$.

One can indicate two values of the exponent α for which the solution in the variables (p, φ) has the form of a straight line passing through the points M and N:

$$p = \alpha k \left[1 - 2\left(\nu + \frac{2}{k}\right)\varphi\right] = \xi \frac{d\varphi}{d\xi},$$

which gives

$$\varphi(\xi) = \frac{1}{2(\nu + 2/k)}\left[1 - \xi^{-2\alpha(k\nu+2)}\right].$$

For the value $\alpha_1 = \dfrac{1}{k\nu + 2}$ we obtain the inverse of the known solution (3).

The second value $\alpha_2 = \dfrac{1}{2(k+1)}$ gives an analogue of solution (4). For $\nu = 2$ both these solutions coincide. The two indicated solutions do not satisfy the boundedness condition for $F(r, 0)$ as $r \neq \infty$, and therefore are not limiting solutions.

Fig. 2

Fig. 3

Since at the instant $\tau = 0$, $F(r, 0) = \text{const} \cdot r^{2-1/\alpha}$ depends on the coordinate in a power-law manner, the function $F(r, t)$ for $t > 0$ is found by a method analogous to that indicated in (5).

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Note: Figure translations are in progress. See original paper for figures.