ASTRONOMY

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1960

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Abstract

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ON GRAVITATIONAL INSTABILITY IN FLAT ROTATING SYSTEMS WITH AXIAL SYMMETRY

(Presented by Academician L. I. Sedov on 31 VIII 1959)

The generalization of the classical Jeans condition for gravitational instability (^{1}) to a rotating medium was carried out by Chandrasekhar (^{2}). An infinite homogeneous medium was considered, rotating about some axis z with an angular velocity independent of the distance from the axis. Chandrasekhar found that the Jeans criterion continues to remain valid for perturbations propagating in all directions except those strictly perpendicular to the axis of rotation. In the case of perturbations perpendicular to the axis, gravitational instability arises only if the density is greater than the critical value

$$\rho > \rho_{\mathrm{cr}} = \frac{\omega^2}{\pi G}. \tag{1}$$

However, this result is inapplicable to real rotating systems, which are usually very flat. In them there can be no instability along z, and it is meaningful to speak only of the propagation of perturbations and the occurrence of instability in the plane perpendicular to the axis of rotation, i.e., precisely where the Jeans criterion is inapplicable. But even for this case Chandrasekhar' s quantitative estimate (1) needs revision. First, because in real systems the rotation velocity is not constant but depends on r, if only because matter is usually concentrated toward the center; and, second, because the size of the system in the direction of the z-axis is considerably smaller than its dimensions in directions perpendicular to z. The physical unreality of systems infinitely extended in the direction of the axis of rotation is emphasized, in particular, by K. F. Ogorodnikov (^{3}).

A closer approximation to reality was made by Bel and Schatzman (^{4}). They assume that the rotation velocity of the system depends on r. Perturbations are considered that propagate in the plane perpendicular to the axis of rotation and are symmetric with respect to this axis. However, the authors considered only the two-dimensional case, and the Poisson equation for the potential is written in the form

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\delta\varphi\right)\right] = -4\pi G\delta\rho. \tag{2}$$

In essence this means that here, too, a system infinitely extended along the z-axis is considered. Since the force of attraction of a flat ring is considerably less than the attraction of the corresponding infinite annular cylinder, one should expect that the condition for gravitational instability obtained by the authors,

$$4\pi G\rho > \frac{2\omega}{r}(\omega r^2)' + \frac{4\pi^2 c^2}{\lambda^2} + \frac{c^2}{4r^2} \tag{3}$$

gives an underestimated value of the critical density.

The problem of the present article is to find the value of the critical density for a real flat rotating cloud. As did the previous authors, we shall assume that the axial symmetry of the cloud is preserved at all times and, consequently, that the perturbations are radial (ring-shaped). Examination of condition (3) shows that it represents the balance of the forces acting on an element displaced as a result of a wave perturbation in the radial direction by an amount $\delta r = 1$, without changing its angular momentum about the center of the system. The term on the left characterizes the attraction of the cylindrical ring in accordance with Poisson' s equation (2); the first term on the right is the force returning the displaced element to its former orbit and associated with the stability of circular orbits; the second term on the right is the gradient of the gas pressure that arises as a result of the perturbation, i.e., also a force returning the element back; the last term is very small in comparison with the preceding ones and may be neglected.

In passing from a system infinite in z to flat systems, only the term in the left-hand side of inequality (3) that is associated with gravitation changes. To determine the component of the gravitational force along r of a ring with density $\delta\rho$, it is now inconvenient to use Poisson' s equation, since an additional term $d^2\delta\varphi/dz^2$ will enter the left-hand side of (2). It is therefore more appropriate to calculate δF_r directly.

Let r be the distance from the axis of rotation, and h the distance from the central plane of the cloud. The r-component of the gravitational force at the point r_0 in the central plane, produced by the perturbation $\delta\rho$, is equal to

$$\delta F_r = G\int_{r_0-\lambda/4}^{r_0+\lambda/4}\int_{-h_0}^{+h_0}\int_0^{2\pi}\frac{\delta\rho\,(r\cos\varphi - r_0)\,r\,dr\,dh\,d\varphi}{\left(r^2 + r_0^2 + h^2 - 2rr_0\cos\varphi\right)^{3/2}}. \tag{4}$$

Integration of (4) with respect to φ leads to elliptic integrals of the first and second kind,

$$\delta F_r = 4Gr_0 \int_{-y_0}^{+y_0} \int_0^{z_0} \frac{\delta\rho(1+y)}{\left[(2+y)^2+z^2\right]^{1/2}} \left[\frac{2y+y^2+z^2}{y^2+z^2} E - K\right] dy\, dz, \tag{5}$$

where

$$y = \frac{r-r_0}{r_0}, \qquad y_0 = \frac{\lambda}{4r_0}, \qquad z = \frac{h}{r_0}, \qquad z_0 = \frac{h_0}{r_0},$$

$$K = \int_0^{\pi/2} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}}, \qquad E = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\psi}\, d\psi, \qquad k = \frac{4(1+y)}{(2+y)^2+z^2}.$$

For a perturbation wavelength λ and half-thickness of the layer h_0 small in comparison with the distance r_0 from the axis of rotation, the quantities y and z are small, and k is close to unity. The elliptic integrals and the other functions entering (5) may be expanded in series and restricted to the first terms[5]:

$$E = 1 + \frac{1}{2}\left(\Lambda - \frac{1}{2}\right) {k'}^2 + \dots, \qquad K = \Lambda + \frac{\Lambda - 1}{4} {k'}^2 + \dots,$$

where ${k'}^2 = 1 - k^2$, $\Lambda = \ln \frac{4}{k'}$. To an accuracy up to small quantities of second order y^2, z^2,

$$E \approx 1 + \frac{1}{16}\left[6\ln 2 - 1 - \ln(y^2+z^2)\right](y^2+z^2),$$

$$K \approx 3\ln 2 - \frac{1}{2}\ln(y^2+z^2) + \frac{1}{16}\left[3\ln 2 - 1 - \frac{1}{2}\ln(y^2+z^2)\right](y^2+z^2).$$

$$\frac{1+y}{\sqrt{(2+y)^2+z^2}} \approx \frac{1}{2}\left(1 + \frac{y}{2} - \frac{y^2}{4} - \frac{z^2}{8}\right).$$

Then the integrand in (5) will be equal to

$$\begin{aligned} \delta\rho f(y,z) = \frac{\delta\rho}{2}\left(1 + \frac{y}{2} - \frac{y^2}{4} - \frac{z^2}{8}\right)\Bigg\{1 - 3\ln 2 + \frac{1}{2}\ln(y^2+z^2) + \\ + \frac{1}{16}(y^2+z^2)\left[3\ln 2 - \frac{1}{2}\ln(y^2+z^2)\right]\Bigg\} + \\ + \frac{2y}{y^2+z^2}\left[1 + \frac{1}{8}\left\{3\ln 2 - \frac{1}{2} - \frac{1}{2}\ln(y^2+z^2)\right\}(y^2+z^2)\right]. \end{aligned} \tag{6}$$

Since $\delta\rho(-y) = -\delta\rho(y)$, the sum of the values of (6) at the points $+y$ and $-y$ becomes the difference $f(y,z) - f(-y,z)$. Therefore

$$\delta F_r = 4Gr_0 \int_0^{y_0} \int_0^{z_0} \delta\rho \left\{ \frac{2y}{y^2+z^2} + \right.$$

$$\left. + y\left[\frac{1}{8} - \frac{3}{4}\ln 2 + \frac{1}{8}\ln(y^2+z^2) - \frac{y^2}{4(y^2+z^2)}\right]\right\} dy\,dz. \tag{7}$$

The second term on the right is small compared with the first, and it may be neglected. We integrate (7) with respect to z, taking outside the integral sign the value of $\delta\rho$ averaged over z:

$$\delta F_r = 8Gr_0 \int_0^{y_0} \delta\rho \operatorname{arc\,tg} \frac{z_0}{y}\, dy. \tag{8}$$

In the case of an ordinary sinusoidal perturbation with maximum displacement δr at the point r_0, and with constant layer thickness $2h_0$,

$$\delta\rho \simeq \left[\frac{2\pi\rho}{\lambda}\sin\frac{\pi y}{2y_0} - \frac{\rho}{r_0(1+y)}\cos\frac{\pi y}{2y_0}\right]\delta r. \tag{9}$$

For $\lambda \ll r_0$ the second term may also be neglected. Then

$$\delta F_r = 4\pi G\rho f(\xi)\,\delta r, \tag{10}$$

where

$$f(\xi) = \int_0^1 \sin\frac{\pi x}{2} \operatorname{arc\,ctg}\frac{\xi x}{2}\, dx, \qquad \xi = \frac{\lambda}{H}. \tag{11}$$

The condition for gravitational instability (3) can then be written in the form

$$4\pi G\rho f(\xi) > \frac{2\omega}{r}(\omega r^2)' + \frac{4\pi^2 c^2}{\lambda^2}. \tag{12}$$

The function $f(\xi)$ has the following values:

ξ	0.2	2	4	6	8	10	14	20
$f(\xi)$	0.96	0.64	0.43	0.34	0.28	0.23	0.172	0.124

It is seen from this that the correction to the value of the critical density turns out to be significant and depends on the ratio of the wavelength of the perturbation to the layer thickness.

Let us now find the value of ξ for which the critical density required for gravitational instability is minimal. We shall use the relation between the layer thickness H and the density ρ_0 in its central plane, found by E. L. Ruskoll$^{(6)}$:

$$H = \sqrt{\frac{2RT}{\pi G \mu \rho_0}}\, I, \tag{13}$$

where

$$I = \frac{1}{2}\int_0^1 \frac{du}{\sqrt{1 - u - \frac{1}{3}u^* \ln u}}, \qquad u^* = \frac{\rho^*}{\rho_0}; \tag{14}$$

ρ^* is the density that would be obtained if the mass of the central body were uniformly distributed inside a sphere of radius r. Tabulation of this integral leads to the following dependence of I on ρ_0/ρ^*:

ρ_0/ρ^*	1/3	1/3	10/3	5	10
I	0.66	0.87	0.94	0.96	0.975

From (13) we obtain

$$\frac{4\pi^2 c^2}{\lambda^2} = 4\pi G\, \frac{\gamma \pi^2 \rho_0}{2I^2\xi^2}. \tag{15}$$

For a system whose rotation is determined mainly by the attraction of the central mass (the Solar System, the outer parts of the Galaxy),

$$\omega r^2 = \sqrt{GMr}, \qquad \frac{2\omega}{r}(\omega r^2)' = \omega^2 = \frac{4}{3}\pi G \rho^*. \tag{16}$$

Then the instability condition (12) may be written in the form

$$\rho > f^{-1}(\xi)\left(\frac{\rho^*}{3} + \frac{\pi^2\gamma}{2I^2}\frac{\rho_0}{\xi^2}\right). \tag{17}$$

The quantity ρ represents the density averaged over z, and therefore is less than ρ_0. But, integrating (7) only over a layer of thickness $2h_0 = H$, which does not contain all the matter of the cloud, we have also somewhat reduced δF_r in (8).

To take into account the attraction of the whole layer, the value of ρ in (17) must be increased somewhat. Numerical estimates show that $\rho \simeq 0.9\rho_0$ and depends little on ξ.

Taking this into account and using the numerical values for $f(\xi)$ and I given above, one can find the critical value of ρ_0 satisfying the instability condition (17). The results of the computations for $\gamma = 1$ give:

ξ	4	6	8	10	15
$\rho_{0\mathrm{cr}}/\rho^*$	6.8	2.3	2.1	2.2	2.4

Thus, the critical density required for gravitational instability, which, as is known, depends on λ, turns out to be minimal at a perturbation wavelength equal to eight times the cloud thickness H. When λ is decreased, its growth is caused by the increase of the second term in (17), associated with the usual Jeans criterion. When λ is increased, the first term, associated with the rotation of the system, begins to play the main role in (17). The critical density then increases as a result of the increase of $f^{-1}(\xi)$, which shows how many times the attraction of a plane ring is less than the attraction of a cylindrical ring infinitely extended along z. The minimum value of the critical density, $\rho_{\mathrm{cr}} = 2.1\rho^*$, is more than 6 times greater than the critical density $\frac{1}{3}\rho^*$ obtained by Bel and Schatzman for the two-dimensional case, and also greater than the value found by Chandrasekhar. Thus, the conditions for gravitational instability in the interstellar matter of the Galaxy prove to be more stringent than was obtained earlier.

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Received
14 VIII 1959

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