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Abstract

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COLLAPSING SYSTEMS OF LINEAR INEQUALITIES

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In the present article, an arbitrary, generally speaking, system of linear inequalities of rank $r > 0$ is studied with the aid of another system of linear inequalities (called a collapse of the given system) having rank less than r and consistent whenever the given system is consistent; moreover, the algorithm of successive collapsing of a system of linear inequalities that follows from this idea is applied to the solution of the basic problems of so-called linear programming.

1. Let L be an arbitrary real linear (vector) space; let $f_1(x), f_2(x), \dots, f_m(x)$ be certain linear (homogeneous linear) functions given on it with real values, and let $a_1, a_2, \dots, a_m$ be certain real numbers. The system of inequalities

$$f_j(x) - a_j \leqslant 0 \quad (j = 1, 2, \dots, m) \tag{1}$$

will be called **consistent** if in the space L there exists a vector $x = x_0$ satisfying all its inequalities. The **rank** of the system (1) will mean the rank of the system of functions $f_j(x)$ entering into it.

If R is the maximal subspace of L on which all the functions $f_j(x)$ vanish (we shall call it the **kernel** of the system (1)), S is any direct complement of it in L ($R + S = L$), $r > 0$ is the rank of the system (1), and $x_1, x_2, \dots, x_r$ is some basis of the space S, then the system

$$t_1 f_j(x_1) + t_2 f_j(x_2) + \dots + t_r f_j(x_r) - a_j \leqslant 0 \quad (j = 1, 2, \dots, m), \tag{2}$$

where $t_1, t_2, \dots, t_r$ are unknowns taking real values, will be called the **frame system** of the system (1).

A certain solution of the system (1) of rank $r > 0$ will be called its **nodal** solution if from it one can select such a subsystem of rank r consisting of r inequalities, all inequalities of which turn into equalities for this solution.

The inequality $f(x) - a \leqslant 0$, where $f(x)$ is a certain linear function given on L with real values and a is a certain real number, is called a **consequence** of the consistent system (1) if it is satisfied by all its solutions.

Replacing the system (1) by one of its frame systems and using theorems 8 and 9 from the work ([1]), one can obtain the following assertion.

Theorem 1. *If the inequality $f(x) - a \leqslant 0$ is a consequence of the consistent system (1) of rank $r > 0$, then it will also be a consequence of at least one such subsystem of it of rank r consisting of r inequalities, all nodal solutions of which satisfy the system (1).*

Corollary 1. *The assertion of the theorem is valid in the case when the consequence of the system (1) is the strict inequality $f(x) - a < 0$.*

Corollary 2. *The inequality $f(x) - a \leqslant 0$ is a consequence of the consistent system (1) of rank $r > 0$ if and only if there exists such a subsystem of it*

$$f_{j_k}(x) - a_{j_k} \leqslant 0 \quad (k = 1, 2, \dots, r)$$

of rank r, that for some r nonnegative numbers $p_1, p_2, \dots, p_r$ the two relations hold:

$$f(x) = p_1 f_{j_1}(x) + p_2 f_{j_2}(x) + \dots + p_r f_{j_r}(x) \quad \text{(identically on } L\text{)},$$

$$p_1 a_{j_1} + p_2 a_{j_2} + \dots + p_r a_{j_r} \leq a.$$

Corollary 3. The assertion of Corollary 2 remains valid if in it the inequalities $f(x) - a \leq 0$ and $p_1 a_{j_1} + \dots + p_r a_{j_r} \leq a$ are replaced by the strict inequalities $f(x) - a < 0$ and $p_1 a_{j_1} + \dots + p_r a_{j_r} < a$.

Corollary 4. A system (1) of rank r is consistent if every subsystem of it of rank r consisting of $r + 1$ inequalities is consistent.

2. We shall call the system (1) of rank $r < m$ **positively intertwined** if, for an arbitrary identically zero positive linear combination (i.e., a linear combination with nonnegative coefficients, some of which are nonzero)

$$p_1 f_{j_1}(x) + p_2 f_{j_2}(x) + \dots + p_{r+1} f_{j_{r+1}}(x), \tag{3}$$

embracing a system of rank r of $r + 1$ functions $f_j(x)$, there corresponds the relation

$$p_1 a_{j_1} + p_2 a_{j_2} + \dots + p_{r+1} a_{j_{r+1}} \geq 0.$$

We shall call a system simply **intertwined** if it is positively intertwined or if from the functions $f_j(x)$ entering it one cannot form even a single such identically zero combination.

If from the functions $f_j(x)$ entering the system (1) of rank r one cannot form even a single such identically zero positive combination (3) for which the equality

$$p_1 a_{j_1} + \dots + p_{r+1} a_{j_{r+1}} = 0$$

would hold, then we shall call the system (1) **stably intertwined**.

We shall call the system (1) **stable** if it is consistent and remains consistent when the equality sign is omitted in its inequalities.

Theorem 2. Every intertwined system (1) is consistent. Every stably intertwined system (1) is stable.

This is proved with the aid of Theorem 1 and its corollaries.

Remark. The propositions converse to the propositions of Theorem 2 are, obviously, valid.

Corollary 1. If from the functions $f_j(x)$ of the system (1) of rank r one cannot form even a single identically zero positive combination (3), then this system is stable.

Corollary 2. A system of homogeneous inequalities $f_j(x) \leq 0$ $(j = 1, 2, \dots, m)$ of rank r is stable if and only if from its functions one cannot form even a single identically zero positive combination (3).

Corollary 3. A system (1) of rank r is stable if every subsystem of it of rank r consisting of $r + 1$ inequalities is stable.

Theorem 3. A system (1) is stable if and only if the set of its solutions contains some base of the space L.

This is proved with the aid of the skeleton system for the system (1).

3. Let U be an arbitrary subspace of L, and let V be some direct complement of it in L. Using the decomposition $x = u + v$, where $u \in U$ and $v \in V$, the system (1) can be written in the form

$$f_j(u) + f_j(v) - a_j \leq 0 \quad (j = 1, 2, \dots, m). \tag{4}$$

Let s be the rank of the system of functions $f_j(u)$, considered on the subspace U; it coincides, obviously, with the dimension of the direct complement of the intersection $R \cup U$ (R is the kernel of the system (1)) in U. If from the system of functions $f_{j_1}(u), f_{j_2}(u), \dots, f_{j_{s+1}}(u)$ of rank s one can form an identically (on U) zero positive linear combination, then we single out

a certain combination of them $t_1 f_{j_1}(u) + \dots + t_{s+1} f_{j_{s+1}}(u)$ of this kind, and form the inequality

$$t_1 f_{j_1}(x) + \dots + t_{s+1} f_{j_{s+1}}(x) - (t_1 a_{j_1} + \dots + t_{s+1} a_{j_{s+1}}) \leqslant 0.$$

The system of all inequalities thus obtained will be called the U-contraction of system (1). If there does not exist any combination of interest to us, then we shall say that the U-contraction of system (1) is empty.

Theorem 4. *Let U be some subspace of L not contained in the kernel R of system (1). If the U-contraction of system (1) is nonempty, then its rank is* *lower than the rank of system (1) by no less than the dimension of the direct complement R^* of the intersection $R \cap U$ in U.*

Let $L = U_1 + U_2 + \dots + U_k + R$ $(k > 1)$, and let S_1 be the U_1-contraction of system (1) and S_n $(n = 1, 2, \dots, k)$ the U_n-contraction of the contraction S_{n-1}. Then the contraction S_k is either empty or identical (i.e., is a system with functions identically equal to zero); here we stipulate that the contraction of an empty contraction is empty. The contractions $S_2, \dots, S_k$ will be called **repeated** contractions of system (1); the contractions considered above will be called **simple**. A repeated contraction S_k will be called **complete**; a simple U-contraction will be called complete if $L = U + R$.

Every nonempty contraction (simple or repeated) of a consistent system is, obviously, consistent. The converse assertion is given by

Theorem 5. *System (1) is consistent if at least one of its simple contractions is consistent or empty.*

Corollary. *System (1) is consistent if at least one of its complete (simple or repeated) contractions is consistent or empty.*

Let U be some subspace of L, and V some direct complement of it in L. If $x = u + v$ $(u \in U,\ v \in V)$ is an arbitrary element of L, then the elements u and v are called the **projections** of the element x, respectively, onto the subspaces U and V.

Theorem 6. *If some U-contraction of system (1) is empty, then the projection of the set M of solutions of this system onto an arbitrary direct complement V of U in L coincides with V. If the U-contraction under consideration is nonempty, then the set of its solutions in the space V coincides with the projection of the set M onto this space.*

4. Let now L be the space R^n of vectors $(x_1, x_2, \dots, x_n)$ over the field R of real numbers, and let $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, \dots, 0), \dots, e_n = (0, 0, \dots, 1)$ be its coordinate basis. Then system (1) takes the form

$$f_j(x) - a_j = a_{j1}x_1 + \dots + a_{jn}x_n - a_j \leqslant 0 \qquad (j = 1, 2, \dots, m), \tag{5}$$

where $a_{ji} = f_j(e_i)$ and $x_1, x_2, \dots, x_n$ are the unknowns.

If U is the subspace spanned in R^n by the vectors $e_{i_1}, \dots, e_{i_k}$, then $f_j(u) = a_{ji_1}x_{i_1} + \dots + a_{ji_k}x_{i_k}$; the U-contraction of system (5) for this chosen U will be called the $(x_{i_1}, \dots, x_{i_k})$-contraction or, otherwise, the contraction with respect to the aggregate of unknowns $x_{i_1}, \dots, x_{i_k}$. It follows from Theorem 5 that system (5) is consistent if it has a nonempty consistent x_i-contraction for at least one of the unknowns x_i.

This proposition has found application in the solution of systems of linear inequalities (in another form it is given in the article ([2])). A method of solving such systems based on it was developed by V. G. Kuznetsov.

In practical applications of so-called linear programming, two problems are of primary importance.

Problem 1. Find the greatest (least) value of the linear form $b_1x_1 + b_2x_2 + \dots + b_nx_n$ on the set of solutions of some consistent system (5).

Problem 2. Indicate at least one solution of the system (5) for which this value is attained.

A brief description of the method proposed here for solving Problem 1, formulated with respect to the system (1) and the linear function $f(x)$ ($x \in L$), is given by

Theorem 7. *If the linear function $f(x)$ ($x \in L$) has a greatest value on the set M of solutions of the consistent system (1), then every complete (simple or repeated) convolution of the system*

$$f_j(x) - a_j \leqslant 0 \quad (j = 1, 2, \dots, m),$$

$$-f(x) + t \leqslant 0 \tag{6}$$

contains the parameter t (i.e., in it at least one of the coefficients of t is different from zero). If some complete (simple or repeated) convolution of the system (6) contains the parameter t, then the function $f(x)$ has a greatest value on the set M. This value is the greatest of the values of the parameter satisfying this convolution.

We now pass to Problem 2, formulated for the system (5). The general case of Problem 2 for the system (1) is reduced, obviously, to this case, if instead of the system (1) one takes some skeleton system (2) of it, and instead of the function $f(x)$ the linear form $t_1f(x_1) + t_2f(x_2) + \dots + t_nf(x_n)$.

Let, for finding the greatest value of the form $b_1x_1 + \dots + b_nx_n$ (denote it by t^*) on the set M^* of solutions of the system (5), there have been constructed some sequence $S_1, S_2, \dots, S_l$ of convolutions (generally speaking, repeated) of the system

$$a_{j1}x_1 + \dots + a_{jn}x_n - a_j \leqslant 0 \quad (j = 1, 2, \dots, m),$$

$$-b_1x_1 - \dots - b_nx_n + t \leqslant 0, \tag{7}$$

ending with the complete convolution S_l. Using Theorem 4, it is not difficult to establish that the number l does not exceed the rank r of the system (5); if $l > 1$, then $S_2, \dots, S_l$ are repeated convolutions.

Having found from the system S_l the greatest value $t = t^*$ which interests us, substitute it (when $l > 1$) into all preceding convolutions S_i; the convolution S_i

for $t = t^*$ will be denoted by S_i^*. By Theorem 5, all the systems thus obtained, as well as the system (7) for $t = t^*$, are consistent.

Let h_1 be an arbitrary solution of the system S_{l-1}^* (when $l = 1$, h_1 is a solution of the system (7) for $t = t^*$). Substituting it into the system S_{l-2}^*, we find some solution h_2 of it (its existence follows from Theorem 5). We substitute it into the system S_{l-3}^*, and so on. Continuing these computations, we finally obtain some solution $h_l = (x_1^0, \dots, x_n^0)$ of the system (7) for $t = t^*$. Obviously,

$$b_1 x_1^0 + \cdots + b_n x_n^0 = t^*.$$

Consequently, h_l is a solution of Problem 2.

In the actual carrying out of these computations, as S_1 one should take some x_i- or (x_i, x_j)-convolution of the system (7), as S_2 some x_i- or (x_i, x_j)-convolution of the system S_1, and so on.

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References Cited

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[2] H. W. Kuhn, *Am. Math. Monthly*, **63**, No. 4, 217 (1956).

Note: Figure translations are in progress. See original paper for figures.