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Abstract

Full Text

MATHEMATICS

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ON TWO APPROXIMATE METHODS FOR SOLVING OPERATOR EQUATIONS IN HILBERT SPACE

(Presented by Academician S. L. Sobolev on 21 XI 1959)

1. Consider the equation

$$Ay = b, \tag{1}$$

where A is a linear self-adjoint positive-definite bounded operator acting in the real Hilbert space H; b is a given element and y is the unknown element of the space H. We denote the upper and lower bounds of the spectrum of the operator A respectively by M and m $(0 < m \leq M)$. The exact solution of equation (1) will be denoted by y^*. Below two approximate methods are proposed for solving equation (1), close in their nature to the method of steepest descent (1). These methods, in our view, are of interest because, simultaneously with the approximate construction of the solution of equation (1), approximate values of one—and, under some additional restrictions, even two—eigenvalues of the operator A are found.

2. Let y_0 be an arbitrary element of the space H. Construct a sequence of elements y_k by the recurrence formulas

$$y_{k+1} = y_k - \frac{2}{\gamma_k + 2\mu_k} x_k, \tag{2}$$

where

$$x_k = Ay_k - b, \qquad \mu_k = \frac{(Ax_k, x_k)}{(x_k, x_k)}, \qquad \gamma_k = \frac{(A\Delta_k, \Delta_k)}{(\Delta_k, \Delta_k)}, \qquad \Delta_k = Ax_k - \mu_k x_k. \tag{3}$$

The process (2), (3) becomes indeterminate if, for some k, the vector Δ_k vanishes. In this case, obviously, x_k is an eigenvector of the operator A, and it is easy to see that

$$y^* = y_k - \frac{1}{\mu_k} x_k.$$

Therefore the case of interest is that in which $\|\Delta_k\| \neq 0$ for all $k = 0, 1, 2, \ldots$.

Theorem 1. *The sequence of elements y_k, obtained from formulas (2), (3), converges in the norm of the space H to the solution y^* of equation (1).*

An exact estimate of the rate of convergence of the process (2), (3) could not be found. We can only assert that the estimate

$$\|y - y^*\| \leq q^k \|y_0 - y^*\|,$$

is valid, where

$$q = \left[1 - \frac{256 m^4 M^3}{(M+m)^5 (M+2m)^2}\right]^{1/2}.$$

We shall need below the simple inequalities

$$\frac{(x,x)^2}{(Ax,x)(A^{-1}x,x)} \geqslant \frac{4mM}{(M+m)^2}, \tag{4}$$

$$\frac{(A^2x,x)^3}{(A^3x,x)^2(x,x)} \geqslant \frac{64m^3M^3}{(M+m)^6}, \tag{5}$$

$$\frac{(A^2x,x)^2}{(A^3x,x)} - (Ax,x) \leqslant 0. \tag{6}$$

The proof of inequality (4) is given, for example, in (1); (5) is easily established with the aid of (4); (6) is a consequence of the inequality for moments (see, for example, (2))

$$(A^s x, x)^{p+r} \leqslant (A^{s+r}x, x)^p (A^{s-p}x, x)^r.$$

We give a brief proof of the theorem. Let $r_k = y_k - y^*$. Then

$$x_k = Ar_k. \tag{7}$$

Subtracting y^* from both sides of relation (2), we obtain

$$r_{k+1} = r_k - \frac{2}{\gamma_k + 2\mu_k} Ar_k, \tag{8}$$

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whence

$$\begin{aligned}\|r_{k+1}\|^2 &= \|r_k\|^2 - \frac{4}{\gamma_k + 2\mu_k}(Ar_k, r_k) + \frac{4}{(\gamma_k + 2\mu_k)^2}(A^2 r_k, r_k)\\ &= \|r_k\|^2 - \frac{4}{\gamma_k + 2\mu_k}(Ar_k, r_k) + \frac{4}{(\gamma_k + 2\mu_k)\mu_k}(A^2 r_k, r_k)\\ &\quad - \frac{4}{(\gamma_k + 2\mu_k)\mu_k}(A^2 r_k, r_k) + \frac{4}{(\gamma_k + 2\mu_k)^2}(A^2 r_k, r_k),\end{aligned}$$

and since

$$\mu_k = \frac{(Ax_k, x_k)}{(x_k, x_k)} = \frac{(A^3 r_k, r_k)}{(A^2 r_k, r_k)}, \tag{9}$$

then, by virtue of (6),

$$\begin{aligned}&-\frac{4}{\gamma_k + 2\mu_k}(Ar_k, r_k) + \frac{4}{(\gamma_k + 2\mu_k)\mu_k}(A^2 r_k, r_k)\\ &= \frac{4}{\gamma_k + 2\mu_k}\left[-(Ar_k, r_k) + \frac{(A^2 r_k, r_k)^2}{(A^3 r_k, r_k)}\right] \leqslant 0.\end{aligned}$$

Consequently,

$$\begin{aligned}\|r_{k+1}\|^2 &\leqslant \|r_k\|^2 - \frac{4}{(\gamma_k + 2\mu_k)\mu_k}(A^2 r_k, r_k) + \frac{4}{(\gamma_k + 2\mu_k)^2}(A^2 r_k, r_k)\\ &= \left[1 - \frac{4(\gamma_k + \mu_k)}{(\gamma_k + 2\mu_k)^2}\,\frac{(A^2 r_k, r_k)}{\mu_k (r_k, r_k)}\right]\|r_k\|^2.\end{aligned}$$

Further, by virtue of (5) and (9),

$$\frac{4(\gamma_k + \mu_k)^2 (A^2 r_k, r_k)}{(\gamma_k + 2\mu_k)^2 \mu_k (r_k, r_k)} = \frac{4(\gamma_k + \mu_k)\mu_k}{(\gamma_k + 2\mu_k)^2}\,\frac{(A^2 r_k, r_k)^3}{(A^3 r_k, r_k)^2 (r_k, r_k)} \geqslant \frac{256 m^3 M^3}{(M+m)^6}\,\frac{(\gamma_k + \mu_k)\mu_k}{(\gamma_k + 2\mu_k)^2}.$$

Moreover,

$$\frac{\gamma_k}{\mu_k} \leqslant \frac{M}{m}, \qquad \frac{(\gamma_k + \mu_k)\mu_k}{(\gamma_k + 2\mu_k)^2} = \frac{\gamma_k/\mu_k + 1}{(\gamma_k/\mu_k + 2)^2} \geqslant \frac{M/m + 1}{(M/m + 2)^2} = \frac{(M+m)m}{(M+2m)^2}.$$

Thus,

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$$\frac{4(\gamma_k + \mu_k)}{(\gamma_k + 2\mu_k)^2} \frac{(A^2 r_k, r_k)}{\mu_k (r_k, r_k)} \geq \frac{256 m^4 M^3}{(M+2m)^2 (M+m)^5},$$

$$\|r_{k+1}\|^2 \leq \left[1 - \frac{256 m^4 M^3}{(M+m)^5 (M+2m)^2}\right] \|r_k\|^2.$$

The theorem is proved.

3. Applying the operator A to (8) and taking (7) into account, for the residuals x_k we obtain

$$x_{k+1} = x_k - \frac{2}{\gamma_k + 2\mu_k} A x_k, \qquad x_0 = A y_0 - b. \tag{10}$$

The iterative process defined by relations (10), up to normalization, coincides with one of the processes for the approximate determination of the lower bound of the spectrum of the operator A, proposed by M. A. Krasnosel' skii in (^{3}), if the initial approximation in the latter is chosen equal to $Ay_0 - b$. The process mentioned was studied by B. P. Pugachev in (^{4}). Using his results, we arrive at the following assertion:

Theorem 2. *Let E_λ be the spectral function of the operator A and*

$$\|E_{m+\varepsilon} x_0\| = \|E_{m+\varepsilon}(Ay_0 - b)\| > 0$$

for every $\varepsilon > 0$. Then

$$\lim_{k\to\infty} \mu_k = m; \qquad \lim_{k\to\infty} \frac{\|A_k^{nx} - m_k^{nx}\|}{\|x_k\|} = 0.$$

Under some additional restrictions on the spectrum of the operator A, one may assert convergence of the sequence of numbers γ_k to the upper bound of the spectrum M. Because of its cumbersomeness we do not give the exact formulation of the result here.

4. The behavior of the residual x_k can be used to obtain a new (in a certain sense better) estimate of convergence of the process (2), (3).

Theorem 3. *For any $\varepsilon > 0$ one can indicate a number k_0 such that for all $p > 0$ the inequality*

$$\|y_{k_0+p} - y^*\| \leq (q+\varepsilon)^p \|y_{k_0} - y^*\|,$$

holds, where

$$q = \frac{M}{M+2m}.$$

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5. In the case when the operator A is only positive, i.e. when $m = 0$, Theorem 1, generally speaking, loses its force. However, in this case one can give the usual weakening of similar theorems. As is known (see, for example, (^{1})), the problem of solving equation (1) reduces to the problem of finding a minimum point of the functional

$$F(x) = (Ax, x) - 2(b, x). \tag{11}$$

Theorem 4. *Let $m = 0$, but equation (1) have a solution y^*. Then the sequence of approximations y_k, obtained by means of the process (2), (3), is a minimizing sequence for the functional (11).*

6. M. A. Krasnosel' skii proposed in (^{3}) several methods for finding the lower bound of the spectrum of the operator A, different from those mentioned above. One of them, called in (^{3}) method δ, was also studied by B. P. Pugachev (^{5}). One can construct an iterative process for the approximate solution of equation (1), in which the residuals $Ay_k - b$, up to normalization, coincide with the sequence of approxi-

obtained by means of procedure δ, when the initial approximation in the latter is chosen to be $Ay_0 - b$. Such a process is described by the formulas

$$y_{k+1} = y_k - \frac{1}{\gamma_k + \mu_k} x_k, \tag{12}$$

where

$$x_k = Ay_k - b, \qquad \mu_k = \frac{(Ax_k, x_k)}{(x_k, x_k)}, \qquad \gamma_k = \frac{(A\Delta_k, \Delta_k)}{(\Delta_k, \Delta_k)}, \qquad \Delta_k = Ax_k - \mu_k x_k. \tag{13}$$

The initial approximation y_0 is chosen arbitrarily.

Theorem 5. *The process (12), (13) converges to the solution y^* of equation (1) at the rate of a geometric progression.*

An exact estimate of the speed of convergence could not be found in this case either. However, one may assert the validity of the estimate

$$\|y_k - y^*\| \leq q^k \|y_0 - y^*\|,$$

where

$$q = \left[1 - \frac{64(2M + m)m^4 M^3}{(M + m)^8}\right]^{1/2}.$$

For the residual $x_k = Ay_k - b$, Theorem 2 is valid. For the process (12), (13), the assertion of item 5 remains in force and a theorem analogous to Theorem 3 holds.

Theorem 6. *For any $\varepsilon > 0$ one can indicate a number k_0 such that, for all $p > 0$, the estimate holds*

$$\|y_{k_0+p} - y^*\| \leq (q+\varepsilon)^p \|y_{k_0} - y^*\|,$$

where

$$q = \frac{M}{M+m}.$$

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Note: Figure translations are in progress. See original paper for figures.

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