MOTION OF AN AXISYMMETRIC JET OF GAS WITH LOW CONDUCTIVITY IN AN AXISYMMETRIC MAGNETIC FIELD

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Abstract

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MOTION OF AN AXISYMMETRIC JET OF GAS WITH LOW CONDUCTIVITY IN AN AXISYMMETRIC MAGNETIC FIELD

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Consider the axisymmetric motion of a compressible, perfect, conducting gas in an axisymmetric magnetic field. We shall neglect the viscosity and thermal conductivity of the gas, and shall regard the coefficient of conductivity of the gas as a constant quantity. We shall also assume that the gas is neutral on the average and that there is no electric field. Introduce cylindrical coordinates $(\tilde{z}, \tilde{r})$, directing the $\tilde{z}$-axis along the axis of symmetry and denoting by $\tilde{r}$ the distance from the axis of symmetry.

Under the assumptions made, the equations of hydrodynamics may be written in the following dimensionless form:

the equation of continuity

$$\frac{\partial}{\partial r}(\rho r v_r) + \frac{\partial}{\partial z}(\rho r v_z) = 0; \quad (1)$$

the equations of motion

$$\rho v_r \frac{\partial v_r}{\partial r} + \rho v_z \frac{\partial v_r}{\partial z} + \frac{\partial p}{\partial r} = \delta H_z (v_z H_r - v_r H_z), \quad (2)$$

$$\rho v_r \frac{\partial v_z}{\partial r} + \rho v_z \frac{\partial v_z}{\partial z} + \frac{\partial p}{\partial z} = -\delta H_r (v_z H_r - v_r H_z); \quad (3)$$

the heat-supply equation

$$\frac{1}{\chi - 1}\left(v_r \frac{\partial p}{\partial r} + v_z \frac{\partial p}{\partial z}\right) - \frac{\chi}{\chi - 1}\frac{p}{\rho}\left(v_r \frac{\partial \rho}{\partial r} + v_z \frac{\partial \rho}{\partial z}\right) = \delta (v_z H_r - v_r H_z)^2.$$

Here r and z are dimensionless coordinates, referred to the characteristic linear dimension a; ρ is the dimensionless density, referred to the characteristic density ρ_{00}; v_r and v_z are the projections of the velocity on the axes r and z, referred to

V_{00}; p is the pressure divided by the characteristic dynamic pressure $\rho_{00}V_{00}^2$; H_z and H_r are the projections of the magnetic-field intensity vector $\mathbf{H}$ on the axes z and r, referred to the characteristic magnetic-field intensity H_{00}; χ is the ratio of specific heats, and δ is a dimensionless parameter, equal to $\delta = \sigma H_{00}^2 a/c^2\rho_{00}V_{00}$, where σ is the coefficient of conductivity and c is the speed of light in vacuum.

In the case of steady motion of a nonviscous, non-heat-conducting gas, in the absence of volume charge and electric field, the equations of motion and heat supply have one first integral—the energy integral—which in the case under consideration is written as follows:

$$\frac{v_r^2 + v_z^2}{2} + \frac{\chi}{\chi - 1}\frac{p}{\rho} = \text{const}, \tag{4}$$

where the constant may be different for different streamlines. This relation may be used instead of the heat-supply equation.

To the system of equations of hydrodynamics one must add the equations of the magnetic field (see, for example, (1)), which, under the assumptions made above, take the form

$$\operatorname{rot}\mathbf{H} = 4\pi\delta\frac{\rho_{00}V_{00}^2}{H_{00}}\,[\mathbf{vH}], \qquad \operatorname{div}\mathbf{H} = 0. \tag{5}$$

Here $\mathbf{v}$ and $\mathbf{H}$ are the velocity and magnetic-field intensity vectors, referred respectively to V_{00} and H_{00}.

If the conductivity of the gas is small and the magnetic field is not very strong, then δ is small compared with unity. Thus, for example, for mercury with characteristic linear dimension $a = 10$ cm and $V_{00} \sim 10$ m/sec: for $H_{00} \sim 500$ gauss, $\delta \sim 2 \cdot 10^{-3}$; for $H_{00} \sim 5000$ gauss, $\delta \sim 0.2$. We shall also assume that the parameter $\rho_{00}V_{00}^2/H_{00}^2$ is of order unity or smaller (i.e., we shall assume that the gas velocities are not very large). In this case all the unknown quantities can be expanded in series in powers of the small parameter δ:

$$\begin{gathered} v_r = v_{r0} + v_{r1}\delta + \dots, \qquad v_z = v_{z0} + v_{z1}\delta + \dots, \qquad p = p_0 + p_1\delta + \dots, \\ \rho = \rho_0 + \rho_1\delta + \dots, \qquad H_z = H_{z0} + H_{z1}\delta + \dots, \qquad H_r = H_{r0} + H_{r1}\delta + \dots. \end{gathered} \tag{6}$$

When the conductivity σ of the gas or liquid is equal to zero, then $\delta = 0$. In this case there is no interaction between the moving gas and the magnetic field. Consequently, $v_{r0}, v_{z0}, p_0, \rho_0$ are the values of the gas parameters without allowance for the influence of the magnetic field, while H_{r0} and H_{z0} are the values of the components of the prescribed external magnetic field without allowance for the additional magnetic fields arising as a result of the interaction of the moving gas and the magnetic field.

The quantities v_{r1}, v_{z1}, etc., make it possible, in the first approximation, to take into account the influence of the magnetic field on the motion of the gas. Substituting expressions (6) into equations (1)–(4) and equating coefficients at like powers of δ, we obtain for the zeroth-order terms the usual equations of hydrodynamics, and for the first-order terms we obtain a system of linear nonhomogeneous equations, whose coefficients will be functions of the zeroth-order quantities. It is essential that only H_{z0} and H_{r0} enter the expression for the magnetic forces in the equations for the first-order quantities. Thus the first-order quantities can be found by regarding the magnetic field as prescribed, independent of the motion of the gas.

Let us consider in more detail the particular case in which, in the absence of a magnetic field, the motion of the gas is the flow of a cylindrical jet of radius r_0 in a space with constant pressure p_{00}, the velocity of the jet being constant and equal to V_{00}, and the density ρ_{00}.

In this case $v_{z0} = 1,\ v_{r0} = 0,\ p_0 = \text{const},\ \rho_0 = 1$, and the equations for the first-approximation quantities take the form:

$$\begin{gathered} \frac{\partial v_{r1}}{\partial r} + \frac{v_{r1}}{r} + \frac{\partial v_{z1}}{\partial z} + \frac{\partial \rho_1}{\partial z} = 0, \\ \frac{\partial v_{r1}}{\partial z} + \frac{\partial p_1}{\partial r} = H_{r0} H_{z0}, \\ \frac{\partial v_{z1}}{\partial z} + \frac{\partial p_1}{\partial z} = -H_{r0}^2, \\ v_{z1} + \frac{\kappa}{\kappa - 1}(p_1 - p_0 \rho_1) = 0. \end{gathered} \tag{7}$$

The quantities sought must evidently satisfy the following boundary conditions.

Since at $z = -\infty$ the flow parameters are constant and equal to the parameters of the undisturbed flow, we have

$$\text{for } z = -\infty \qquad v_{z1} = 0, \qquad v_{r1} = 0, \qquad p_1 = 0, \qquad \rho_1 = 0. \tag{8}$$

At the boundary of the jet, for $r = r_{\text{gr}}$, the pressure is constant and, consequently, $p_1 = 0$. Since r_{gr} itself can be represented in the form

$$r_{\text{gr}} = r_0 + \delta r_{\text{gr}1} + \cdots,$$

then, to within terms of order δ^2, fulfillment of the condition $p_1 = 0$ may be required not at $r = r_{\text{gr}}$, but at $r = r_0$, i.e.

$$\text{for } r = r_0 \qquad p_1 = 0, \tag{9}$$

Integrating the third equation of the system (7), using the boundary condition (8), one can express from it and from the energy integral ρ_1 and p_1 in terms of v_{z1}:

$$p_1 = -v_{z1} - \int_{-\infty}^{z} H_{r0}^2 \, dz, \qquad \rho_1 = -M_{00}^2 v_{z1} - \varkappa M_{00}^2 \int_{-\infty}^{z} H_{r0}^2 \, dz, \tag{10}$$

where $M_{00} = \rho_{00} V_{00}^2 / \varkappa p_{00}$ is the square of the Mach number of the unperturbed flow.

Eliminating p_1 and ρ_1 from the first two equations (7), we obtain

$$\frac{\partial v_{r1}}{\partial r} + \frac{v_{r1}}{r} + (1 - M_{00}^2)\frac{\partial v_{z1}}{\partial z} = \varkappa M_{00}^2 H_{r0}^2,$$

$$\frac{\partial v_{r1}}{\partial z} - \frac{\partial v_{z1}}{\partial r} = H_{r0} H_{z0} + \int_{-\infty}^{z} \frac{\partial H_{r0}^2}{\partial r} \, dz. \tag{11}$$

It is easy to see that the solution of the problem can be reduced to the integration of the following linear differential equation of second order:

$$\frac{\partial^2 v_{r1}}{\partial r^2} + \frac{\partial}{\partial r}\frac{v_{r1}}{r} + (1 - M_{00}^2)\frac{\partial^2 v_{r1}}{\partial z^2}$$

$$= (1 - M_{00}^2)\left[\frac{\partial}{\partial z}(H_{r0} H_{z0}) + \frac{\partial}{\partial r} H_{r0}^2\right] + \varkappa M_{00}^2 \frac{\partial}{\partial r} H_{r0}^2. \tag{12}$$

The boundary conditions for the function v_{r1} will be:

$$\text{for } z = -\infty \qquad v_{r1} = 0;$$

$$\text{for } r = r_0 \qquad \frac{\partial v_{r1}}{\partial r} + \frac{v_{r1}}{r} = \left[1 + (\varkappa - 1) M_{00}^2\right] H_{r0}^2$$

(the last condition is obtained from the condition $p_1 = 0$ for $r = r_0$ and equations (10) and (11)).

This equation is easily integrated in the case $M_{00} = 1$. Its solution satisfying the boundary conditions will be

$$v_{r1} = \frac{\varkappa}{r} \int_0^r r H_{r0}^2 \, dr.$$

From equations (11) and the boundary conditions we obtain

$$v_{z1} = -\varkappa \int_r^{r_0} \frac{dr}{r} \int_0^r r \frac{\partial H_{r0}^2}{\partial r}\, dr + \int_r^{r_0} H_{r0} H_{z0}\, dr - \int_{-\infty}^z H_{r0}^2\, dr.$$

Having found v_{z1} from formulas (10), we determine ρ_1 and p_1.

In the general case, finding an exact solution of equation (12) is difficult; therefore we shall consider in more detail the case of a thin jet, when the radius of the jet is small in comparison with the characteristic linear dimension a, i.e. r_0 is small in comparison with unity. We expand v_{r1} and v_{z1} in series in powers of r:

$$v_{r1} = \sum_{i=0}^{\infty} v_{r1i}(z) r^i, \qquad v_{z1} = \sum_{i=0}^{\infty} v_{z1i}(z) r^i.$$

Suppose that the magnetic field has no singularities on the axis of symmetry. Since H_{r0} and H_{z0} satisfy the equations $\operatorname{rot} \mathbf{H}_0 = 0$ and $\operatorname{div} \mathbf{H}_0 = 0$, it can be shown that they may be represented in the form

$$H_{r0} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\varphi^{(2n+1)}}{2^{2n+1} n!(n+1)!} r^{2n+1}, \qquad H_{z0} = \sum_{n=0}^{\infty} (-1)^n \frac{\varphi^{(2n)}}{2^{2n} (n!)^2} r^{2n},$$

where $\varphi(z)$ is an arbitrary function, obviously equal to the value of H_{z0} on the axis of symmetry. Substituting the series expansions for v_{r1}, v_{z1}, H_{r0}, and H_{z0} into equations (11) and equating the coefficients of like powers of r, we obtain a system of equations that allows v_{ri} and v_{zi} to be expressed in terms of one arbitrary function z, which is determined from the boundary condition. Carrying out the calculations, for example, to terms of order r_0^4, we have

$$v_r = \delta \left[\frac{1 - M_{00}^2}{8} (\varphi'^2 + \varphi\varphi'') r_0^2 r - \left(\frac{1 - M_{00}^2}{16} \varphi\varphi'' - \frac{\chi M_{00}^2}{16} \varphi'^2 \right) r^3 \right],$$

$$v_z = 1 - \frac{\delta}{4} \left[\left(\int_{-\infty}^z \varphi'^2\, dz \right) r_0^2 + \left(\int_{-\infty}^z \varphi\varphi''\, dz \right) (r_0^2 - r^2) \right].$$

From formulas (10), p_1 and ρ_1 can readily be found.

The shape of the jet can be determined from the condition of conservation of flow rate through the jet. In dimensionless variables this condition is written as follows:

$$r_0^2 = 2 \int_0^{r_{\text{gr}}} r \rho v_z\, dr.$$

From it, to terms of order r_0^6, we find:

$$r_{\text{gr}}^2 = r_0^2 \left\{ 1 + \frac{\delta r_0^2}{8} \left[(1 - M_{00}^2) \int_{-\infty}^{z} (2\varphi'^2 + \varphi\varphi'')\, dz + \chi M_{00}^2 \int_{-\infty}^{z} \varphi'^2\, dz \right] \right\}.$$

Let us consider the shape of the jet for a specific magnetic field produced by a circular current of radius a, situated in the section $z = 0$. In this case (see, for example, (1)),

$$H_{z0} = H_{00}(1 + z^2)^{-3/2}, \qquad \varphi(z) = (1 + z^2)^{-3/2}.$$

If we denote $\Delta s = 8(r_{\text{gr}}^2 - r_0^2)/\delta r_0^4$, it is easy to compute that the derivative Δs vanishes at $z = \pm\infty$ and at $z = \pm z_0$, where

$$z_0 = \sqrt{\frac{1 - M_{00}^2}{10 + (3\chi - 10) M_{00}^2}}.$$

It follows readily from this that for $M_{00} < 1$ the jet expands from $z = -\infty$ to $z = -z_0$, then contracts between the values $-z_0$ and z_0, and expands again between $z = z_0$ and $z = +\infty$.

For $M_{00} > 1$, two cases are possible:

1. $3\chi > 10$ and $M_{00} > 1$ arbitrary, or $3\chi < 10$ and $1 < M_{00} < \sqrt{10/(10 - 3\chi)}$. In these cases the jet expands from $z = -\infty$ to $z = +\infty$ (z_0 is imaginary).
2. $3\chi < 10$, $M_{00} > \sqrt{10/(10 - 3\chi)}$. In this case the jet contracts between $z = -\infty$ and $z = -z_0$, expands for $-z_0 < z < z_0$, and contracts again for $z > z_0$.

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REFERENCES

1. L. D. Landau, E. M. Lifshitz, *Electrodynamics of Continuous Media*, Moscow, 1957.

Note: Figure translations are in progress. See original paper for figures.