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Abstract

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ON THE MOTION OF A GYROSCOPIC COMPASS UNDER THE ACTION OF RANDOM FORCES

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The accuracy of a gyroscopic compass with mercury ballistic vessels under conditions of a ship' s rolling can be ensured, as was shown by Rawlings ([1]), by a proper choice of the period of the natural oscillations of the mercury in the ballistic vessels. Rawlings' theory proceeds from the assumption that the ship' s rolling takes place according to a sinusoidal law. In the present paper the motion of a gyroscopic compass is studied under the assumption that the ship' s rolling is a stationary random process ([2]) with fractional-rational spectral density.

1. The equations of motion of a gyroscopic compass with mercury ballistic vessels during the rolling of a ship may be written in the form ([1])

$$H\beta' + HU\cos\varphi \cdot \alpha - c_1\vartheta + c\vartheta = Hv_N/R, \qquad H\alpha' + c\vartheta = -HU\sin\varphi,$$

$$C\vartheta'' + m\vartheta' + \vartheta + \beta = -W_2/g, \qquad A\gamma'' + s\gamma' + lP\gamma = lPW_1/g. \tag{1}$$

Here α is the angle of rotation of the gyrocompass in azimuth; β is the angle of elevation of the axis of the gyroscope rotor above the horizontal plane; ϑ is the angle between the mercury mirror in the ballistic vessels and the axis of the gyroscope rotor; γ is the angle of rotation of the cardan suspension of the gyrocompass about an axis parallel to the axis of the gyroscope rotor; H is the kinetic moment of the gyroscope; U is the angular velocity of the diurnal rotation of the terrestrial sphere; φ is the latitude of the place of observation; v_N is the northern component of the ship' s velocity; R is the radius of the terrestrial sphere; W_1 and W_2 are the eastern and northern components of the translational acceleration of the support point of the gyrocompass.

Under conditions of rectilinear uniform motion of the ship,

$$W_1 \simeq -r\theta''\cos\psi, \qquad W_2 \simeq r\theta''\sin\psi, \tag{2}$$

where θ is the ship' s roll angle; ψ is the ship' s course; r is the distance of the support point of the gyrocompass from the straight line passing through the center of the ship' s rolling and parallel to the ship' s longitudinal axis.

Introduce new variables $X_1, \dots, X_4$ by means of the relations

$$X_1 = \alpha - \alpha^*, \qquad X_2 = \beta - \beta^*, \qquad X_3 = \vartheta - \vartheta^*, \qquad X_4 = \gamma - \gamma^*, \qquad (3)$$

where $\alpha^* = v_N/RU\cos\varphi - \lambda\tan\varphi,\ \beta^* = -\vartheta^* = HU\sin\varphi/c,\ \gamma^* = 0,\ \lambda = c_1/c.$ Then equations (1) are transformed to the form

$$X_1' + \frac{k^2}{U\cos\varphi}X_3 = 0,$$

$$X_2' + U\cos\varphi\, X_1 - \lambda\frac{k^2}{U\cos\varphi}X_3 - U\sin\varphi\, X_4 + \frac{k^2}{U\cos\varphi}X_3X_4 = 0,$$

$$X_3'' + \zeta X_3' + n^2X_3 + n^2X_2 = -\frac{r\sin\psi}{g}\, n^2\theta'', \qquad (4)$$

$$X_4'' + \chi X_4' + \rho^2X_4 = -\frac{r\cos\psi}{g}\, \rho^2\theta'',$$

where $k^2 = cU\cos\varphi/H,\ n^2 = 1/C,\ \rho^2 = lP/A,\ \zeta = m/C,\ \chi = s/A.$

The equations of the first approximation, which are obtained from (4) by discarding terms of second order of smallness, may be represented in the form of the matrix equation

$$f(D)X^{(0)} = e(D)\theta(t) \qquad (D = d/dt), \qquad (5)$$

where

$$f(D) = \begin{vmatrix} D & 0 & \dfrac{k^2}{U\cos\varphi} & 0 \\ U\cos\varphi & D & -\lambda\dfrac{k^2}{U\cos\varphi} & -U\sin\varphi \\ 0 & n^2 & D^2+\zeta D+n^2 & 0 \\ 0 & 0 & 0 & D^2+\chi D+\rho^2 \end{vmatrix}, \qquad X^{(0)} = \begin{Vmatrix} X_1^{(0)} \\ X_2^{(0)} \\ X_3^{(0)} \\ X_4^{(0)} \end{Vmatrix}, \qquad e(D) = \begin{Vmatrix} 0 \\ 0 \\ -\dfrac{r\sin\psi}{g}\, n^2D^2 \\ -\dfrac{r\cos\psi}{g}\, \rho^2D^2 \end{Vmatrix}. \qquad (6)$$

From equation (5) it follows that

$$X^{(0)} = Y(D)\theta(t) \qquad (Y(D) = F(D)e(D)/\Delta(D)). \qquad (7)$$

Here $F(D)$ is the adjugate matrix for the matrix $f(D)$, and $\Delta(D)$ is the determinant of the matrix $f(D)$,

$$\Delta(D) = (D^2 + \chi D + \rho^2)\Delta_1(D), \tag{8}$$

where $\Delta_1(D) = D^4 + \zeta D^3 + n^2 D^2 + \xi D + k^2 n^2, \quad \xi = \lambda n^2 k^2 / U \cos\varphi$.

Below we shall need the quantities $X_3^{(0)}$ and $X_4^{(0)}$. According to (7), they are determined by the operational expressions

$$X_3^{(0)} = \left[-\frac{r\sin\psi}{g}\frac{n^2 D^4}{\Delta_1(D)} + \frac{r\cos\psi}{g}\frac{\rho^2 n^2 U \sin\varphi D^3}{\Delta(D)}\right]\theta(t),$$

$$X_4^{(0)} = -\frac{r\cos\psi}{g}\frac{\rho^2 D^2}{D^2 + \chi D + \rho^2}\theta(t). \tag{9}$$

Let us note that it follows from (9) that

$$X_3^{(0)} = \Phi(D) X_4^{(0)}, \tag{10}$$

where

$$\Phi(D) = \frac{n^2 \tan\psi}{\rho^2}\frac{D^2(D^2 + \chi D + \rho^2)}{\Delta_1(D)} - \frac{n^2 U \sin\varphi D}{\Delta_1(D)}.$$

Assuming that the ship' s rolling is a stationary random process with spectral density

$$S(\omega) = L\frac{4\mu\nu^2}{(\omega^2 - \nu^2)^2 + 4\mu^2\omega^2}, \tag{11}$$

we proceed to determining the mathematical expectations of the random processes $X_1, \dots, X_4$, which we denote as follows:

$$x_i = M[X_i] \qquad (i = 1, \dots, 4). \tag{12}$$

To determine the functions x_i $(i = 1, \dots, 4)$, in accordance with (4), we shall have the system of differential equations

$$\dot{x}_1 + \frac{k^2}{U\cos\varphi}x_3 = 0,$$

$$\dot{x}_2 + U\cos\varphi \cdot x_1 - \lambda\frac{k^2}{U\cos\varphi}x_3 - U\sin\varphi \cdot x_4 = -\frac{k^2}{U\cos\varphi}E,$$

$$\ddot{x}_3 + \zeta \dot{x}_3 + n^2 x_3 + n^2 x_2 = 0,$$

$$\ddot{x}_4 + \chi \dot{x}_4 + \rho^2 x_4 = 0, \tag{13}$$

where $E = M[X_3 X_4]$.

Restricting ourselves to the first approximation, we shall replace E in equations (13) by the expression

$$E_0 = M\left[X_3^{(0)} X_4^{(0)}\right]. \tag{14}$$

The integrals of the system of homogeneous equations that is obtained from (13) for $E = 0$ will tend asymptotically to zero as $t \to \infty$, since the zeros of the characteristic determinant $\Delta(D)$, according to (8), are located in the left half-plane of the complex variable D. Therefore, during the rolling of the ship, which is a long-duration process, the quantities $x_1, \dots, x_4$ will assume their steady-state values

$$x_1 = -\left(k/U\cos\varphi\right)^2 E_0, \qquad x_2^* = 0, \qquad x_3^* = 0, \qquad x_4^* = 0. \tag{15}$$

Here

$$E_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{43}(\omega)\, d\omega, \tag{16}$$

where $S_{43}(\omega)$ is the cross-spectral density of the random processes $X_4^{(0)}$ and $X_3^{(0)}$,

$$S_{43}(\omega) = \Phi(i\omega) S_{44}(\omega), \tag{17}$$

and $S_{44}(\omega)$ is the spectral density of the random process $X_4^{(0)}$,

$$S_{44}(\omega) = 4\mu\nu^2 L \frac{\rho^4 r^2 \cos^2\psi}{g^2} \frac{\omega^4}{\left[(\omega^2-\rho^2)^2+\chi^2\omega^2\right]\left[(\omega^2-\nu^2)^2+4\mu^2\omega^2\right]}. \tag{18}$$

Expression (16) can be reduced to the form

$$E_0 = 4\mu\nu^2 L(z_1 I + z_2 J), \tag{19}$$

where

$$z_1 = \frac{1}{2g^2}\rho^2 n^2 r^2 \sin 2\psi, \qquad z_2 = \frac{1}{g^2}\rho^4 n^2 U \sin\varphi\, r^2 \cos^2\psi; \tag{20}$$

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{g_1(i\omega)}{h(i\omega)h(-i\omega)}\, d\omega, \qquad J = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{g_2(i\omega)}{h(i\omega)h(-i\omega)}\, d\omega; \tag{21}$$

$$g_1(i\omega) = \sum_{s=0}^{7} b_s (i\omega)^{14-2s}, \qquad g_2(i\omega) = \sum_{s=0}^{7} B_s (i\omega)^{14-2s},$$

$$h(i\omega) = \sum_{s=0}^{8} a_s (i\omega)^{8-s}, \tag{22}$$

and the coefficients of the polynomials (22) have the form $b_0 = 0,\ b_1 = 1,\ b_2 = \rho^2 + n^2 - \chi\xi,\ b_3 = (\rho^2 + k^2)n^2 - \chi\xi,\ b_4 = \rho^2 k^2 n^2,\ b_5 = b_6 = b_7 = 0,\ B_0 = B_1 = B_2 = 0,\ B_3 = \zeta,\ B_4 = \xi,\ B_5 = B_6 = B_7 = 0,\ a_0 = 1,\ a_1 = 2\mu + \chi + \zeta$, $a_2 = \nu^2+\rho^2+n^2+2\mu(\chi+\zeta)+\chi\zeta$, $a_3 = 2\mu(\rho^2+n^2+\chi\zeta)+\zeta(\rho^2+\nu^2)+\chi(\nu^2+n^2)+\xi$, $a_4 = 2\mu(\xi + \chi n^2 + \zeta\rho^2) + \chi(\xi + \zeta\nu^2) + n^2(\nu^2 + \rho^2 + k^2) + \nu^2\rho^2$, $a_5 = 2\mu(k^2 n^2 + \rho^2 n^2 + \chi\xi) + \xi(\rho^2 + \nu^2) - \chi n^2(\nu^2 + k^2) + \zeta\nu^2\rho^2$, $a_6 = 2\mu(\xi\rho^2 + \chi k^2 n^2) + n^2(\nu^2\rho^2 + k^2\nu^2 + k^2\rho^2) + \chi\xi\nu^2$, $a_7 = (2\mu\rho^2 + \chi\nu^2)k^2 n^2 + \xi\nu^2\rho^2$, $a_8 = k^2 n^2 \rho^2 \nu^2$.

The integrals (21), in the case when all zeros of the polynomial $h(D)$ are located in the left half-plane of the complex variable D, according to Phillips ([3]), have the form

$$I = -M_1/2a_0 N, \qquad J = -M_2/2a_0 N, \tag{23}$$

where

$$N = \begin{vmatrix} a_1 & a_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \\ 0 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_8 & a_7 & a_6 & a_5 & a_4 \\ 0 & 0 & 0 & 0 & 0 & a_8 & a_7 & a_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_8 \end{vmatrix}, \tag{24}$$

where M_1 and M_2 are determinants obtained by replacing the first column in determinant (24) by a column whose elements are $b_0, b_1, \dots, b_7$ or $B_0, B_1, \dots, B_7$, respectively.

As follows from (15), under ship-rolling conditions the gyroscopic compass has an azimuth deviation. In accordance with (15) and (19), this deviation is determined by the expression

Fig. 1

Figure 1: Fig. 1

$$x_1^* = -aI^* \sin 2\psi - bJ^* \cos^2 \psi, \tag{25}$$

where

$$a = 2\mu\gamma^2 L\rho^2 \frac{r^2}{g^2}\left(\frac{k}{U\cos\varphi}\right)^2, \qquad b = 4\mu\gamma^2 L\rho^4 U \sin\varphi \frac{r^2}{g^2}\left(\frac{k}{U\cos\varphi}\right)^2,$$

$$I^* = n^2 I, \qquad J^* = n^2 J. \tag{26}$$

2. As an example, let us determine the deviation of a gyroscopic compass whose parameters have the values $k = 1.24 \cdot 10^{-3}$ sec^{-1}, $\rho = 6$ sec^{-1}, $\lambda = 1.2$ sec^{-1}, $\zeta = 0.2n$ sec^{-1}, $\xi = 10^{-3}n^2$ sec^{-3}, $r = 2$ m. The latitude of the observation point is $\varphi = 60^\circ$, so that $U\cos\varphi = 3.646 \cdot 10^{-5}$ sec^{-1}. The parameters determining the spectral density of the ship' s rolling are: $\mu = 0.1$ sec^{-1}, $\gamma = 0.8$ sec^{-1}, $\sqrt{L} = 0.173 \simeq 10^\circ$.

Fig. 1

For these data, the deviation of the gyrocompass, according to (25), is

$$x_1^* = -6.72\, I^* \sin 2\psi - 3.03 \cdot 10^{-2} J^* \cos^2 \psi. \tag{27}$$

Here, according to (26) and (23), $I^* = I^*(T)$, $J^* = J^*(T)$, where $T = 2\pi/n$. The graphs of the functions $I^*(T)$ and $J^*(T)$ are given in Fig. 1.

The maximum value of J^* in the example under consideration is 0.0037, so that the second term in expression (27) does not exceed $0.11 \cdot 10^{-3}$, or 0.4 angular minutes. Neglecting this quantity as small, we obtain from (27) the following expression for the deviation of the gyrocompass:

$$x_1^* = -6.72\, I^* \sin 2\psi. \tag{28}$$

The deviation x_1^* is an **intercardinal deviation** of the gyroscopic compass. It vanishes on the cardinal courses of the ship ($\psi = 0;\ 90;\ 180;\ 270^\circ$) and attains its greatest values on the intercardinal courses ($\psi = 45;\ 135;\ 225;\ 315^\circ$).

To prevent the intercardinal deviation, the parameters of mercury ballistic vessels must be chosen so that the value of the angular frequency n of the natural oscillations of the mercury mirror is sufficiently close to the value of the argument n^* at which the function $I^*(n)$ becomes zero. As can be seen from the graph, the zero of the function $I^*(n)$ is located near the point $n = \gamma$. In the example considered, $n^* = 0.843$.

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Note: Figure translations are in progress. See original paper for figures.