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Abstract

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On Some Solutions of a Linear Nonhomogeneous System of Difference Equations with Linear Coefficients

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In the paper (1), A. F. Leont' ev studied analytic solutions of a homogeneous differential-difference equation with linear coefficients. A. F. Leont' ev' s method was developed by A. A. Mirolyubov as applied to a nonhomogeneous equation (2). For a difference equation with constant coefficients, A. G. Naftalevich established the existence of meromorphic solutions having, in a certain strip, arbitrarily prescribed poles (3).

In the present paper it is shown that the main results of the works cited also hold for a nonhomogeneous system of difference equations with linear coefficients.

Consider the system of equations

$$\sum_{j=1}^{m}\sum_{k=1}^{n}(a_{ik}^{(j)}z+b_{ik}^{(j)})y_k(z+h_j)=F_i(z),\qquad i=1,2,\dots,n, \tag{1}$$

where $0=h_1<h_2<\dots<h_m$; a_{ik},b_{ik} are complex numbers.

For brevity we shall agree to say that a system of functions $\{f_1(z),\dots,f_n(z)\}$ is entire, analytic in the half-plane P, etc., if each of the functions of this system is, respectively, entire, analytic in the half-plane P, etc. We shall denote the left-hand side of system (1) briefly by $L\{y_1,\dots,y_n\}$. In this notation system (1) is written as

$$L\{y_1,\dots,y_n\}=\{F_1,\dots,F_n\}. \tag{1'}$$

Introduce the determinants

$$\Delta_1(z)=|a_{ik}^{(1)}z+b_{ik}^{(1)}|,\qquad \Delta_m(z)=|a_{ik}^{(m)}z+b_{ik}^{(m)}|.$$

The role of these determinants is analogous to the role of the lowest and highest coefficients in the case of a single equation (4). Namely, it is not difficult to show that a solution $\{y_1(z),\dots,y_n(z)\}$, analytic in the half-plane $\operatorname{Re} z<a$, can

be analytically continued to the plane with cuts $l_s^{(m)}$ $(s = 1, 2, \dots, n)$, drawn parallel to the positive real semiaxis from the points $k_s^{(m)} + h_m$ $(s = 1, \dots, n)$, where $k_s^{(m)}$ are the roots of $\Delta_m(z)$. In exactly the same way, a solution analytic in the right half-plane can be analytically continued to the plane with cuts $l_s^{(1)}$ $(s = 1, 2, \dots, n)$, drawn parallel to the negative real semiaxis from the points $k_s^{(1)}$ $(s = 1, \dots, n)$, where $k_s^{(1)}$ are the roots of $\Delta_1(z)$.

Introduce also the determinants

$$\Delta_1 = |a_{ik}^{(1)}|, \qquad \Delta_m = |a_{ik}^{(m)}|, \qquad \Delta(t) = |a_{ik}(t)|,$$

where

$$a_{ik}(t) = \sum_{j=1}^{m} a_{ik}^{(j)} e^{h_j t}.$$

Throughout the paper we assume $\Delta_1 \neq 0$ and $\Delta_m \neq 0$.

In addition, we shall assume that $\Delta(t)$ has no real zeros (we can always achieve this by the substitution $y_k(z) = e^{i\varphi z} Y_k(z)$, $k = 1, 2, \dots, n$, where φ is a suitably chosen real number).

The formal scheme of the solution is as follows. Suppose that we can find solutions of the systems

$$L_z\{f_{1\nu}(z,\xi), \dots, f_{n\nu}(z,\xi)\} = \left\{ \overbrace{0, \dots, 0}^{\nu-1}, \frac{1}{\xi - z}, 0, \dots, 0 \right\}, \quad \nu = 1, 2, \dots, n. \quad (2)$$

(L_z means that the operator L is applied with respect to the variable z.) Then the solution of system (1), under certain conditions, will be given by the formulas

$$y_k(z) = \frac{1}{2\pi i} \int_C \sum_{\nu=1}^{n} f_{k\nu}(z,\xi) F_\nu(\xi)\, d\xi, \quad k = 1, 2, \dots, n. \quad (3)$$

We shall seek solutions of the systems (2) in the form

$$f_{k\nu}(z,\xi) = \int_{C_1} u_{k\nu}(t,\xi) e^{zt}\, dt, \quad k, \nu = 1, 2, \dots, n, \quad (4)$$

where the functions $u_{k\nu}(t,\xi)$ and the contour C_1 are to be determined.

To determine the functions $u_{k\nu}(t,\xi)$ we obtain n nonhomogeneous linear systems of ordinary differential equations of the first order. In all these systems the corresponding homogeneous system is one and the same, and its coefficients do not depend on ξ. We denote this homogeneous system symbolically by

$$\frac{du}{dt} = A(t)u. \tag{5}$$

For $\Delta_1 \neq 0,\ \Delta_m \neq 0$, the elements of the matrix $A(t)$ have derivatives summable on the interval $(-\infty, +\infty)$. Therefore, if we assume that the matrix $A(\infty)$ (for $\Delta_1 \neq 0,\ \Delta_m \neq 0$ its elements are finite) has no multiple eigenvalues, then system (5) is reduced by a linear substitution to L-diagonal form (5). If, in addition, we assume that the real parts of the eigenvalues of $A(\infty)$ are distinct, then it follows easily from (5) that system (5) has n particular solutions $\{u_{1\nu}(t), \dots, u_{n\nu}(t)\}$ such that the functions $u_{k\nu}(t)$, as $t \to +\infty$, are asymptotically representable in the form

$$u_{k\nu}(t) = c_{k\nu} t^{\sigma_{k\nu}} e^{\lambda_\nu t}\,[1 + O(t^{-1})], \quad k, \nu = 1, 2, \dots, n.$$

Analogous asymptotic representations can also be obtained for $t \to -\infty$.

Lemma. *There exist real numbers A, B, C, A_1, B_1, C_1 such that, for any k, ν, we have:*

$$\begin{aligned} &\text{for } \operatorname{Re}\xi < A \text{ and } t > 0, \quad |u_{k\nu}(t,\xi)| < Be^{(C - \operatorname{Re}\xi)t};\\ &\text{for } \operatorname{Re}\xi > A_1 \text{ and } t < 0, \quad |u_{k\nu}(t,\xi)| < B_1 e^{(C_1 - \operatorname{Re}\xi)t}. \end{aligned}$$

The final result is similar to the result obtained for one equation with coefficients that are polynomials of degree n (6). Take an arbitrary convex closed contour such that the segment MN, joining the "uppermost" (M) and the "lowermost" (N) points of this contour, is parallel to the imaginary axis. The points M and N divide the contour Γ into two parts; we denote the "right" part by Γ_1, and the "left" part by Γ_2. Shift Γ_1 by h_m "to the right" and denote the new position of Γ_1 by Γ_1^*, and the new positions of the points M and N by M^* and N^*, respectively. From the points M and N draw half-lines parallel to the negative real semi-axis, and from the points M^* and N^* half-lines parallel to the positive real semi-axis. The resulting half-strips (bounded by the drawn half-lines and the arcs Γ_1^* and Γ_2) will be denoted by P_1 and P_2. The union of the half-strips $P_1,\ P_2$ and the rays $l_s^{(1)}, l_s^{(m)}$ $(s = 1, 2, \dots, n)$ (see above) will be called the domain I_Γ.

Theorem 1. If the right-hand side of system (1) is entire, then system (1) has a solution $\{y_1(z), \dots, y_n(z)\}$ satisfying it inside the contour Γ and regular in the domain complementary to I_Γ.

We shall now show that, in addition to the solution constructed, the system under consideration has solutions that have, in the domain complementary to I_Γ, poles, and, moreover, in a certain strip these poles may be prescribed arbitrarily.

Draw two straight lines: through the points M, N and through the points M^*, N^*. We shall regard the resulting vertical strip of width h_m as half-closed:

$$\operatorname{Re} M \leq \operatorname{Re} z < \operatorname{Re} M^*.$$

Denote this strip by π. Let

$$\lambda \in \pi, \qquad \operatorname{Re} M + h_\sigma \le \operatorname{Re}\lambda < \operatorname{Re} M + h_{\sigma+1}.$$

Following A. G. Naftalevich (^{3}), consider the set Q, consisting of the points:

$$1)\ \lambda; \qquad 2)\ \lambda_{k_1\dots k_q} = +\sum_{i=1}^{q}(h_m - h_{k_i}); \qquad 3)\ \mu_{l_1\dots l_q} = \lambda - \sum_{i=1}^{q} h_{l_i}$$

$$(q = 1, 2, \dots;\ k_1 = 1, 2, \dots, \sigma;\ l_1 = \sigma + 1, \dots, m;\ k_2, \dots, k_q = 1, 2, \dots, m - 1;$$
$$l_2, \dots, l_q = 2, 3, \dots, m).$$

Denote

$$\tilde{\lambda}_{k_1\dots k_q} = \lambda_{k_1\dots k_q} - h_m; \qquad \tilde{\mu}_{l_1\dots l_q} = \mu_{l_1\dots l_q} \quad (h_1 = 0).$$

Take in the strip π an arbitrary countable sequence of points

$$\{\lambda^{(r)}\}, \quad r = 1, 2, \dots,$$

having no finite limit points. For each point $\lambda^{(r)}$, construct in the indicated way a set Q_r.

Denote by $\zeta^{(\nu)}$, $\operatorname{Re}\zeta^{(\nu)} \ge \operatorname{Re} M$, and by $\tilde{\xi}^{(\nu)}$, $\operatorname{Re}\tilde{\xi}^{(\nu)} < \operatorname{Re} M$ $(\nu, \tilde{\nu} = 1, 2, \dots)$, all those points at which at least one of the functions on the right-hand side of system (1) has a pole. Put:

$$\xi^{(\nu)}_{k_1\dots k_q} = \zeta^{(\nu)} + \sum_{i=1}^{q}(h_m - h_{k_i}),$$

$$q = 1, 2, \dots; \quad k_1, \dots, k_q = 1, 2, \dots, m - 1; \quad \nu = 1, 2, \dots;$$

$$\tilde{\tilde{\xi}}^{(\tilde{\nu})}_{l_1\dots l_q} = \tilde{\xi}^{(\tilde{\nu})} - \sum_{i=1}^{q} h_{l_i}, \quad q = 1, 2, \dots; \quad l_1, \dots, l_q = 2, 3, \dots, m; \quad \tilde{\nu} = 1, 2, \dots.$$

Theorem 2. Let the right-hand side of system (1) be meromorphic (in particular, entire). If none of the points

$$\zeta^{(\nu)}, \quad \xi^{(\nu)}_{k_1\dots k_q}, \quad \tilde{\lambda}^{(r)}_{k_1\dots k_q}$$

is a zero of $\Delta_m(z)$, and none of the points

$$\tilde{\xi}^{(\tilde{\nu})}, \quad \tilde{\tilde{\xi}}^{(\tilde{\nu})}, \quad \tilde{\mu}^{(r)}_{l_1\dots l_q}$$

is a zero of $\Delta_1(z)$, then there exist functions

$$\{f_1(z), \dots, f_n(z)\},$$

which give a solution of system (1) inside the contour Γ and have, at the points $\lambda^{(r)}$, poles with arbitrarily prescribed principal parts

$$R_k^{(r)}(z, \lambda^{(r)}) = \frac{c_{k1}^{(r)}}{z - \lambda^{(r)}} + \dots + \frac{c_{kp_r}^{(r)}}{(z - \lambda^{(r)})^{p_r}}, \qquad k = 1, 2, \dots, n.$$

The other singularities of $\{f_1(z), \dots, f_n(z)\}$ in the strip π can occur only in the intersection of this strip with I_Γ. In the domain complementary to

$$\pi + I_\Gamma,$$

the functions $\{f_1(z), \dots, f_n(z)\}$ have only poles.

A solution of the same character is also possessed by the homogeneous system.

The results of the first part of the work carry over to differential-difference systems. The results of the second part can be carried over only to particular types of differential-difference systems, for example to systems of the form

$$\sum_{k=1}^{n} \left[(a_{ik}^{(1)} z + b_{ik}^{(1)})\, y_k(z + h_1) + (a_{ik}^{(m)} z + b_{ik}^{(m)})\, y_k(z + h_m) \right] +$$

$$+ \sum_{j=2}^{m-1} \sum_{k=1}^{n} \sum_{\nu=0}^{p} (a_{ik\nu}^{(j)} z + b_{ik\nu}^{(j)})\, y^{(\nu)}(z + h_j) = F_i(z), \qquad i = 1, 2, \dots, n.$$

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