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Abstract

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ON CONFORMAL MAPPINGS OF SPACE

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Liouville' s theorem on conformal mappings of space is usually proved for mappings of class C^3 (see, for example, ([1,2])). From the results of M. A. Lavrent' ev ([3]) it follows that it is valid in the class $C^{1,\alpha}$, $0 < \alpha < 1$. In the present paper a proof is given that is free of any assumptions concerning differentiability of the mapping.

In what follows: E^n is n-dimensional Euclidean space; $|\mathbf{x}|$ is the norm of the vector $\mathbf{x} \in E^n$; $Q(\mathbf{x}_0, r)$ is the ball $\{|\mathbf{x} - \mathbf{x}_0| < r\}$; $S(\mathbf{x}_0, r)$ is the sphere $\{|\mathbf{x} - \mathbf{x}_0| = r\}$; $W_p^l(M)$, where $M \subset E^n$ is an open set, is the set of all functions defined in M having generalized partial derivatives of order l in the sense of S. L. Sobolev ([4]), summable with power p on every compact set $F \subset M$.

For an arbitrary measurable function $\lambda(x) \geq 0$, put

$$F_\lambda(\mathbf{x}, r) = \int_{S(\mathbf{x},r)} [\lambda(\mathbf{x})]^{(n-1)/2}\, d\sigma,$$

where $d\sigma$ is the element of area of the sphere $S(\mathbf{x}, r)$,

$$V_\lambda(\mathbf{x}, r) = \iint_{Q(\mathbf{x},r)} [\lambda(\mathbf{x})]^{n/2}\, d\mathbf{x}, \qquad d\mathbf{x} = dx_1\, dx_2 \ldots dx_n.$$

If $\mathbf{x} \in M \subset E^n$, then let $r_0(\mathbf{x}, M)$ be the distance from $\mathbf{x}$ to the boundary of M. Let $\mathbf{x}$ be an arbitrary point in E^n. $U_t(\mathbf{x})$, where t is a real number, $0 < t < 1$, is a family of neighborhoods of the point $\mathbf{x}$. The family $U_t(\mathbf{x})$ is called **regular** if: a) $U_t(\mathbf{x}) \subset U_1(\mathbf{x})$ for all t, and there exists a topological mapping $\varphi(x)$ of $U_1(\mathbf{x})$ into E^n such that $\varphi[U_t(\mathbf{x})] = Q(0, t)$, $0 < t < 1$; b) if $R(t)$ and $r(t)$ are the least and the greatest, respectively, of the distances from the point $\mathbf{x}$ to the points of the boundary of $U_t(\mathbf{x})$, then as $t \to 0$, $R(t)/r(t) \to 1$.

Let $M \subset E^n$ be an open set. A mapping $\mathbf{y}(\mathbf{x})$ of M into E^n is called **conformal** if: a) $\mathbf{y}(\mathbf{x})$ is a topological mapping; b) for every point $\mathbf{x} \in M$ there is a regular family of neighborhoods which, under the mapping $\mathbf{y}(\mathbf{x})$, passes into a regular family of neighborhoods of the point $\mathbf{y}(\mathbf{x})$.

Theorem. *Every conformal mapping in E^n for $n \geq 3$ is a combination of a finite number of inversions.*

To prove the theorem it suffices to establish the analyticity of a conformal mapping. The proof is based on Lemmas 1-7. Fix some domain M and a conformal mapping $\mathbf{y}(\mathbf{x})$ of M into E^n.

Lemma 1. *The vector-function $\mathbf{y}(\mathbf{x}) \in W_n^1(M)$. Its partial derivatives $\partial \mathbf{y}/\partial x_i$, for almost all $\mathbf{x} \in M$, are mutually orthogonal and have equal lengths.*

Set $\lambda(x) = (\partial y/\partial x_j)^2$.

Lemma 2. For every measurable set $E \subset M$, the set $y(E)$ is measurable and its Lebesgue measure is equal to

$$\iint_E [\lambda(x)]^{n/2}\, dx, \qquad dx = dx_1 \cdots dx_n.$$

Lemma 3. For every point $x \in M$, for almost all r, $F_\lambda(x, r)$ is equal to the area of the surface $y[S(x, r)]$ (area is understood in the sense of Lebesgue).

Applying the isoperimetric inequality in E^n, we obtain the following lemma:

Lemma 4. For each point $x \in M$, for all $r \in [0, r_0(x)]$, the inequality

$$[F_\lambda(x, r)]^n \geq n^{n-1} \omega_{n-1} [V_\lambda(x, r)]^{n-1},$$

holds, where ω_{n-1} is the area of the unit sphere in E^n.

Let $\alpha_h(Z)$ be the Sobolev averaging kernel (4). Let M_h be the set of all $x \in M$ for which $r_0(x, M) \geq h$. Put

$$\lambda_h(x) = \left\{ \iint_{|z| \leq h} [\lambda(x+z)]^{n/2} \alpha_h(z)\, dz \right\}^{2/n}.$$

The function $\lambda_h(x)$ is defined and positive in M_h, and there has all derivatives of arbitrary order.

Lemma 5. For every point $x \in M$, if $h < r_0(x)$, for all $r \in [0, r_0(x) - h]$ the isoperimetric inequality

$$[F_{\lambda_h}(x, r)]^n - n^{n-1} \omega_{n-1} [V_{\lambda_h}(x, r)]^{n-1} \geq 0$$

is satisfied.

Lemma 6. Let $\lambda(x) > 0$ be a function defined in M and having continuous derivatives there up to and including the second order. Then for all $x \in M$

$$\lim_{r\to 0}\frac{[F_\lambda(x,r)]^n - n^{n-1}\omega_{n-1}[V_\lambda(x,r)]^{n-1}}{r^2[F_\lambda(x,r)]^n} = \frac{2n-2}{n^2-4}\,\frac{\Delta[\lambda(x)]^{(n-2)/4}}{[\lambda(x)]^{(n-2)/4}},$$

where Δ is the Laplace operator, $n > 2$.

From Lemmas 5 and 6 it follows that, for every point $x \in M$, when $h < r_0(x)$, $\Delta[\lambda_h(x)]^{(n-2)/4} \geq 0$, i.e., the function $[\lambda_h(x)]^{(n-2)/4}$ is subharmonic. Hence, passing to the limit as $h \to 0$, we obtain that the function $[\lambda(x)]^{(n-2)/4}$ is subharmonic. Thus $\lambda(x)$ is bounded above on every compact set $A \subset M$. Taking into account that the mapping inverse to $y(x)$ is also conformal, we obtain that the function $1/\lambda(x)$ is bounded on every compact set $A \subset M$. Therefore, for every compact set $A \subset M$, there exist constants λ_0 and λ_1 such that

$$0 < \lambda_0 \leq \lambda(x) \leq \lambda_1 < \infty$$

for all $x \in A$.

Lemma 7. If a subharmonic function $u(x)$, defined in M, is bounded on every compact set $A \subset M$, then $u(x) \in W_2^1(M)$.

It follows from Lemma 7 that $\lambda(x) \in W_2^1(M)$.

We take the components $y_1, y_2, \dots, y_n$ of the vector $y(x)$ as coordinates in M. The function y_k, $k = 1, 2, \dots, n$, satisfies Laplace' s equation in the coordinates $(y_1, \dots, y_n)$ and, consequently, minimizes the Dirichlet integral for the given boundary values. In the coordinates $(x_1, \dots, x_n)$ this integral

is equal to

$$\iint_{Q(x_0,r)} \sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\right)^2 [\lambda(x)]^{n/2-1}\,dx. \qquad (1)$$

Thus, $y_k(x)$ minimizes the integral (1) for the given boundary values. Hence, $y_k(x)$ is a generalized solution of equation (5)

$$\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left\{[\lambda(x)]^{(n-1)/2}\frac{\partial y_k}{\partial x_i}\right\} = 0.$$

Using the already established properties of the functions $\lambda(x)$ and $y_k(x)$, $k = 1, 2, \dots, n$, it is not difficult from this first to establish that $y_k(x) \in W_2^2(M)$, and then, by the usual methods [5] of the theory of partial differential equations, to prove the analyticity of $y_k(x)$.

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Note: Figure translations are in progress. See original paper for figures.