Reports of the Academy of Sciences of the USSR

MATHEMATICS

1960

SovietRxiv

Abstract

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Reports of the Academy of Sciences of the USSR
1960. Volume 134, No. 2

MATHEMATICS

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A GENERALIZATION OF THE PHRAGMÉN-LINDELÖF THEOREM FROM ANALYTIC FUNCTIONS TO HARMONIC FUNCTIONS IN SPACE

(Presented by Academician M. V. Keldysh on 3 V 1960)

Theorem 1. Let $u(r, \varphi, x)$ be a harmonic function in the cylinder $r \le a,\ 0 \le \varphi < 2\pi,\ -\infty < x < \infty$. If the conditions

$$u(a, \varphi, x) = 0, \qquad \left|\frac{\partial u}{\partial r}(a, \varphi, x)\right| < C; \tag{1}$$

$$\max_{(r,\varphi)} |u(r, \varphi, x)| < C \exp e^{\pi|x|/(2+\varepsilon)a}, \qquad \varepsilon > 0, \tag{2}$$

are satisfied, then $u(r, \varphi, x) \equiv 0$.

Theorem 2. Let $u(r, \theta, \varphi)$ be a harmonic function in the cone $0 < r < \infty,\ 0 \le \varphi < 2\pi,\ 0 \le \theta \le \theta_0 < \pi$. If the conditions

$$u(r, \theta_0, \varphi) = 0, \qquad \left|\frac{\partial u}{\partial \theta}(r, \theta_0, \varphi)\right| < C; \tag{1'}$$

$$\max_{(\theta,\varphi)} |u(r, \theta, \varphi)| < C \exp\left(r + \frac{1}{r}\right)^{\pi/2\theta_0 - \varepsilon}, \qquad \varepsilon > 0, \tag{2'}$$

are satisfied, then $u(r, \theta, \varphi) \equiv 0$.

The proof of these two results is based on the following uniqueness theorem for Dirichlet series:

Theorem 3. Let

$$F(z) = \sum_{n=1}^{\infty} a_n e^{\lambda_n z}$$

be an entire function, and

$$|a_n|^{1/n} < \frac{C}{n^{2+\varepsilon}}, \qquad \varepsilon > 0; \tag{3}$$

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = \alpha, \qquad 0 < \alpha < \infty, \qquad \lambda_n > 0. \tag{4}$$

If, moreover, $|F(x)| < C, \ -\infty < x < \infty$, then $F(z) \equiv 0$.

We begin with the proof of Theorem 3. Put

$$F_\rho(z) = \rho \int_{-\infty}^{\infty} F(t) \exp\left(-e^{\rho(t-z)} + \rho(t-z)\right) \, dt. \tag{5}$$

Lemma 1. If

$$|F(t)| < Ce^{-\delta|t|}, \qquad -\infty < t < \infty, \qquad 0 < \delta < \rho, \tag{6}$$

then $F_\rho(x+iy)$ is regular in the strip

$$-\infty < x < \infty, \qquad |y| < \pi/2\rho - \eta, \qquad \eta > 0, \tag{7}$$

and satisfies in this strip the inequality

$$|F_\rho(x+iy)| < Ce^{-\delta|x|}. \tag{8}$$

Proof. The uniform convergence of (5) in the strip (7) follows from the estimate

$$|F(t)\exp(-e^{\rho(t-x-iy)} + \rho(t-x-iy))| <$$

$$< C\exp(-\delta|t| - e^{\rho(t-x)}\cos\rho y - \rho(t-x)),$$

since there $\cos\rho y > 0$. Let us estimate $F_\rho(x+iy)$. For $x > 0$ we have

$$|F_\rho(x+iy)| < C_\rho \int_{-\infty}^{\infty} \exp(-\delta|u+x| - e^{\rho u}\cos\rho y + \rho u)\, du < Ce^{-\delta x},$$

since $|u+x| \geq u+x$. The estimate for $x < 0$ is analogous.

Lemma 2. If

$$F(z) = \sum_{n=1}^{\infty} a_n e^{\lambda_n z}$$

is an entire function and conditions (3), (4) are satisfied, and $\rho > 1/(2+\varepsilon)\alpha$, then

$$F_\rho(z) = \sum_{n=1}^{\infty} a_n \Gamma\left(\frac{\lambda_n}{\rho} + 1\right) e^{\lambda_n z}. \tag{9}$$

Proof. Put $z = x$ in formula (5) and make the change of variable $u = e^{\rho(t-x)}$, $t = \frac{1}{\rho}\ln u + x$. Then we obtain

$$F_\rho(x) = \int_0^{\infty} \left(\sum_{n=1}^{\infty} a_n e^{\lambda_n x} u^{\lambda_n/\rho}\right) e^{-u}\, du.$$

Applying the theorem on termwise integration (see (1), Ch. I, § 7.9), we obtain that $F_\rho(z)$ is representable by the Dirichlet series (9), provided this series converges. The condition for convergence of the series (9) has the form $\rho \geq 1/(2+\varepsilon_1)\alpha$, $0 < \varepsilon_1 < \varepsilon$, since, when it is fulfilled, the coefficients of the series (9) satisfy the inequality

$$|a_n|\Gamma(\lambda_n/\rho + 1) < C \cdot n^{-\varepsilon_2 n}, \qquad \varepsilon_2 > 0.$$

Lemma 3. Let the function $f(t + i\lambda)$ be regular in the strip $|\lambda| \leq \gamma$, $-\infty < t < \infty$, and satisfy there the inequality

$$|f(t + i\lambda)| < Ce^{-\delta|t|}.$$

Then for the function

$$\varphi(z) = \int_{-\infty}^{\infty} f(t)e^{-tz}\, dt,$$

regular in the domain $|\operatorname{Re} z| < \delta$, the estimate

$$|\varphi(iy)| < Ce^{-\gamma|y|}$$

holds.

Proof. Since $f(t + i\lambda) \to 0$ as $t \to \pm\infty$, $|\lambda| \leq \gamma$, we have

$$\varphi(z) = \int_{i\beta-\infty}^{i\beta+\infty} f(\zeta)e^{-z\zeta}\, d\zeta, \qquad |\beta| \leq \gamma,$$

therefore, for $y > 0$,

$$\varphi(iy) = \int_{i\gamma-\infty}^{i\gamma+\infty} f(\zeta)e^{-iy\zeta}\,d\zeta = e^{-\gamma y}\int_{-\infty}^{\infty} f(t-i\gamma)e^{-ity}\,dt,$$

whence $|\varphi(iy)| < Ce^{-\gamma y}$, $y > 0$, and analogously for $y < 0$.

Lemma 4. Denote

$$G_\rho(z) = \prod_{n=1}^{\infty}\left(1-\frac{z^2}{\lambda_n^2}\right)\int_{-\infty}^{\infty} F_\rho(t)e^{-tz}\,dt, \qquad \rho > \frac{1}{(2+\varepsilon)\alpha}. \tag{10}$$

The function $G_\rho(z)$ can be analytically continued into the half-plane $\operatorname{Re} z \geqslant 0$ and satisfies there the inequalities

$$|G_\rho(iy)| < Ce^{\pi|y|\left(\alpha-\frac{1}{2\rho}+\varepsilon_3\right)} \qquad (\varepsilon_3 > 0 \text{ arbitrary}); \tag{11}$$

$$|G_\rho(z)| < Ce^{b|z|}. \tag{12}$$

Proof. Consider the functions

$$\Phi_1(z) = \int_0^{\infty} F_\rho(t)e^{-tz}\,dt, \qquad \Phi_2(z) = \int_{-\infty}^{0} F_\rho(t)e^{-tz}\,dt.$$

Since, by Lemma 1, $|F_\rho(t)| < Ce^{-\delta|t|}$, the function $\Phi_1(z)$ is regular for $\operatorname{Re} z \geqslant 0$, and the integral for $\Phi_2(z)$ when $\operatorname{Re} z < \delta$ can be evaluated by termwise integration. Namely, since $\rho > 1/(2+\varepsilon)\alpha$,

$$\Phi_2(z) = \int_{-\infty}^{0}\left(\sum_{n=1}^{\infty} a_n\Gamma\left(\frac{\lambda_n}{\rho}+1\right)e^{\lambda_n t}\right)e^{-zt}\,dt = \sum_{n=1}^{\infty}\frac{a_n\Gamma\left(\lambda_n/\rho+1\right)}{\lambda_n - z}. \tag{13}$$

The formula obtained gives an analytic continuation of $\Phi_2(z)$ to the whole z-plane. Thus $G_\rho(z)$ is continued into the half-plane $\operatorname{Re} z \geqslant 0$, and from representations (10) and (13) it is clear that $G_\rho(z)$ is regular for $\operatorname{Re} z \geqslant 0$.

We now prove inequality (11). Apply Lemma 3 to the function $F_\rho(t+i\lambda)$. By Lemma 1 we obtain for the parameter γ in Lemma 3 the value $\gamma = \pi/2\rho - \eta$, where $\eta > 0$ is arbitrary. Therefore

$$\left|\int_{-\infty}^{\infty} F_\rho(t)e^{-iyt}\,dt\right| < Ce^{-(\pi/2\rho-\eta)|y|}. \tag{14}$$

For the canonical product

$$\psi(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right), \qquad \lim_{n\to\infty} \frac{n}{\lambda_n} = \alpha,$$

the inequality (see, for example, (2), p. 87)

$$\max_{|z|=y} |\psi(z)| = \psi(\pm iy) \leqslant Ce^{(\pi\alpha+\varepsilon)|y|} \tag{15}$$

holds, where $\varepsilon > 0$ is arbitrary. From (14) and (15), (11) follows.

To prove (12), note that the function $\Phi_2(z)$ is bounded on a sequence of circles $|z| = r_k$, $r_k \to \infty$, $r_{k+1} - r_k < C$, since outside the circles $|z - \lambda_n| < n^{-2}$ we have

$$|\Phi_2(z)| < C \sum_{n=1}^{\infty} n^{-\varepsilon n} n^2 < C,$$

and the sum of the diameters of the circles inside which this inequality is not valid is less than a constant. This observation and inequality (15) prove (12).

Proof of Theorem 3. Note that without loss of generality we may assume that $F(x)$ satisfies inequality (6), since otherwise we could take the function $F(z)e^{-\lambda_0 z/2}$, for which all the conditions would be fulfilled.

Now apply Lemmas 1-4 successively, taking $\rho = 1/(2+\varepsilon_1)\alpha$. The constructed function $G_\rho(z)$ is regular in the half-plane $\operatorname{Re} z \geqslant 0$ and satisfies inequalities (11), (12). But from the Phragmén–Lindelöf principle it follows (see (3), Sec. III, Ch. 6, No. 327) that such a $G_\rho(z) \equiv 0$. This, in turn, means that $F_\rho(z) \equiv 0$, i.e. $a_n = 0$, $n = 1, 2, \ldots$.

Proof of Theorem 1. Since $u(r, \varphi, x)$ has in the cylinder $r \leqslant a$, $0 \leqslant \varphi < 2\pi$, $-\infty < x < \infty$ two continuous derivatives and satisfies the boundary condition $u(a, \varphi, x) = 0$, it can be expanded in the series

$$u(r,\varphi,x) = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} e^{in\varphi} J_n\left(\lambda_{n,m}\frac{r}{a}\right) \frac{1}{J_n'(\lambda_{n,m})} \left(a_{n,m}e^{\lambda_{n,m}x/a} + b_{n,m}e^{-\lambda_{n,m}x/a}\right),$$

where $J_n(x)$ are Bessel functions; $\lambda_{n,m}$ is the m-th positive root, in order of magnitude, of $J_n(x) = 0$. Put

$$v(r,\varphi,x) = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \frac{a_{n,m}}{J_n'(\lambda_{n,m})} e^{in\varphi} J_n\left(\lambda_{n,m}\frac{r}{a}\right) e^{\lambda_{n,m}x/a}.$$

It is clear that conditions (1), (2) hold also for $v(r,\varphi,x)$, if only they hold for $u(r,\varphi,x)$. Therefore it suffices to consider $v(r,\varphi,x)$.

By virtue of the boundedness of $\dfrac{\partial v}{\partial r}(a,\varphi,x)$ we have

$$\left|\int_0^{2\pi}\frac{\partial v}{\partial r}(a,\varphi,x)e^{-in\varphi}\,d\varphi\right| < C\left|\sum_{m=1}^{\infty} a_{n,m}\lambda_{n,m}e^{\lambda_{n,m}x/a}\right| < C.$$

Consider the entire function

$$f_n(z)=\sum_{m=1}^{\infty} a_{n,m}\lambda_{n,m}e^{\lambda_{n,m}z}$$

and show that it satisfies all the conditions of Theorem 3.

First, it is well known that

$$\lim_{m\to\infty}\frac{m}{\lambda_{n,m}}=\frac{1}{\pi},$$

i.e. condition (4) is fulfilled and $\alpha = 1/\pi$. Secondly, from condition (2) of Theorem 1 we have

$$\int_0^{2\pi}\int_0^{a}|v(r,\varphi,x)|^2\,r\,dr\,d\varphi=\sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty}|a_{n,m}|^2e^{2\lambda_{n,m}x/a} < C\exp 2\rho x,$$

where $\rho=\pi/(2+\varepsilon)a$. Consequently,

$$|a_{n,m}| < C\min_x \exp\left\{\rho x-\lambda_{n,m}\frac{x}{a}\right\}=C\exp\left[-\frac{\lambda_{n,m}}{a\rho}\left(\ln\frac{\lambda_{n,m}}{a\rho}-1\right)\right],$$

which gives us (for fixed n)

$$(\lambda_{n,m}|a_{n,m}|)^{1/m} < Cm^{-2-\varepsilon}.$$

Thus all the conditions of Theorem 3 are satisfied, $f_n(z)\equiv 0$, and hence also $v(r,\varphi,x)\equiv 0$. The theorem is proved.

The proof of Theorem 2 is carried out in exactly the same way; only, for $u(r,\theta,\varphi)$, one uses the expansion into the series

$$u(r,\theta,\varphi)=\sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty}e^{in\varphi}H_{n,m}(\theta)\left(a_{n,m}r^{\lambda_{n,m}}+b_{n,m}r^{-\lambda_{n,m}-1}\right).$$

Here $\lambda_{n,m}$ are eigenvalues, and $H_{n,m}(\theta)$ are eigenfunctions of the problem

$$H''(\theta)+H'(\theta)\operatorname{ctg}\theta+\left(\lambda(\lambda+1)+\frac{m^2}{\sin^2\theta}\right)H(\theta)=0, \qquad |H(0)|<\infty, \quad H(\theta_0)=0.$$

By refining Theorem 3 one could obtain, in Theorems 1 and 2, the replacement of conditions (2) and (2′) respectively by the conditions

$$\ln\max_{(r,\varphi)}|u(r,\varphi,x)| = o\big(e^{\pi|x|/2a}\big), \qquad x\to\pm\infty,$$

$$\ln\max_{(\theta,\varphi)}|u(r,\theta,\varphi)| = o\left(\left(r+\frac{1}{r}\right)^{\pi/2\theta_0}\right), \qquad r\to 0,\infty.$$

A further strengthening of Theorems 1 and 2 is already impossible. In the case of Theorem 2 and $\theta_0=\pi/2$, this is seen from the example

$$u(r,\theta,\varphi)=\cos(\mu r\sin\theta\cos\varphi)\operatorname{sh}(\mu r\cos\theta) \qquad (\mu>0 \text{ arbitrary}).$$

Theorems 1 and 2 contain the answer to certain questions posed by S. N. Mergelyan in ([4]).

Received
28 IV 1960

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Note: Figure translations are in progress. See original paper for figures.