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Abstract

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ON THE CONNECTION BETWEEN THE LAW OF DECAY AND THE FIRST-ORDER MOMENT OF THE ENERGY DISTRIBUTION

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1. In the well-known work ([1]), the fundamental theorem of Fok–Krylov* in the quantum theory of decay was obtained, concerning the connection between the law of decay $L(t)$ and the energy distribution $\tilde{\omega}(E)$ of a physical system:

$$p_1(t) \equiv M_1(t)e^{iN_1(t)} = \int_{-\infty}^{\infty} \tilde{\omega}(E)e^{iEt}\, dE, \tag{1}$$

where $L(t) = M_1^2(t)$**. It follows from this theorem, in particular, that the law of decay is completely and uniquely determined by the energy distribution of the decaying physical system.

In our previous works ([2–6]), on the basis of explicit use of the spectrality principle (the semi-finiteness of $\tilde{\omega}(E)$), the inverse problem of reconstructing the energy distribution $\tilde{\omega}(E)$ from the law of decay $L(t)$ was posed and studied in detail:

$$\tilde{\omega}(E) = \frac{1}{2\pi} \int_{-\infty}^{\infty} M_1(t) \exp\{-iEt + iN_1(t)\}\, dt. \tag{2}$$

This problem is evidently reduced to determining the phase $N_1(t)$ of the basic function $p_1(t)$*** from its modulus $M_1(t)$, which is connected with the experimentally determined law of decay $L(t) = M_1^2(t)$ ([2–6]). For this purpose dispersion relations ([2–6]) were used, following from analyticity in the upper half-plane $\operatorname{Im} t > 0$. In so doing, the ambiguity in the solution of the inverse problem was singled out, associated with the possible zeros of $p_1(t)$ in the upper half-plane $\operatorname{Im} t > 0$ ([2–6]). General results were obtained concerning the possible location of the zeros of $p_1(t)$; in particular, it was proved that $p_1(t)$ belongs to class A ([4–6]), and also such classes of energy distributions $\tilde{\omega}(E)$ were isolated for which the problem of reconstruction from the experimentally determined law of decay $L(t)$ is unique ([4–6]).

Below new results are obtained in this direction, of interest in connection with the proposal of Matthews and Salam ([10]) to characterize unstable particles by moments of energy distributions.

2. The refinement obtained below of the problem of the connection of the law of decay $L(t)$ with the energy distribution $\tilde{\omega}(E)$ of a physical system concerns energy distributions that decrease sufficiently rapidly as $E \to \infty$

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* Quite recently this result was repeated in the works of M. Lévy ([7]), P. Matthews and A. Salam ([8]), and I. Petzold ([9]).

** It is more convenient for us to consider $p_1(t)$ ([2–6]), and not $p(t)$, introduced in (1); they are related by $p_1(t) \equiv p(-t)$; we also assume $\hbar = 1$.

*** In the terms of probability theory, $p_1(t)$ is a characteristic function.

$\tilde{\omega}(E)$, namely those for which there exists a finite first-order moment. We formulate the result obtained in the form of a theorem.

Theorem. *The decay law $L(t)$ of a physical system whose energy distribution*

$$\tilde{\omega}(E) = \begin{cases} \omega(E) \geqslant 0, & E \geqslant 0, \\ 0, & E < 0, \end{cases} \tag{3}$$

has a finite first-order moment $\tilde{\omega}^{(1)}$

$$0 < \tilde{\omega}^{(1)} \equiv \int_{-\infty}^{\infty} E\tilde{\omega}(E)\, dE < \infty, \tag{4}$$

must necessarily satisfy the relation

$$0 \leqslant \frac{1}{2\pi} \int_0^{\infty} \frac{\ln L(t)}{t^2}\, dt + \frac{1}{2}\tilde{\omega}^{(1)} = \sum_k \frac{\operatorname{Im} t_k}{|t_k|^2}, \tag{5}$$

where the summation over k is the summation over all possible (isolated) zeros t_k of the basic function $p_1(t)$ in the upper half-plane $\operatorname{Im} t_k > 0$.

Without dwelling here on the proof of this theorem*, we emphasize that it rests essentially on the following results: a) the membership of $p_1(t)$, on the basis of the Bochner-Khinchin theorem ([11]), in the positive-definite functions; b) Carleman' s theorem ([12]), and c) the necessary condition imposed on the modulus of the characteristic function $p_1(t)$ (and, consequently, also on the decay probability $L(t)$) ([2–6])**

$$\int_{-\infty}^{\infty} \frac{|\ln|p_1(t)||}{1+t^2}\, dt = \frac{1}{2} \int_{-\infty}^{\infty} \frac{|\ln L(t)|}{1+t^2}\, dt < \infty, \tag{6}$$

which follows from the Paley–Wiener theorem ([13]) on the basis of the spectrality principle (3).

The theorem obtained above is a typical result of the theory of entire functions, giving a connection between the number of zeros, the behavior of the modulus of an analytic function at infinity $t \to \infty$, and (which is essential for us here) at zero $t \sim 0$.

3. Let us consider the main consequences, of physical interest, of the theorem formulated above. From (5) it is obvious that

$$\left| \int_0^{\infty} \frac{\ln L(t)}{t^2}\, dt \right| < \infty. \tag{7}$$

On the basis of the positive definiteness of $p_1(t)$, we obtain the following inequality, which has an obvious physical meaning:

$$M_1(t) \leqslant M_1(0) = 1. \tag{8}$$

Then (7) is equivalent to

$$\int_0^{\infty} \frac{|\ln L(t)|}{t^2}\, dt < \infty. \tag{9}$$

* In ([10]) the mass distribution $\rho(x^2)$ is used, which is connected with the energy distribution in an obvious way.

** In previous papers ([2–6]) this necessary condition was called the criterion of physical realizability in the quantum theory of decay.

It is obvious that the result (9) obtained includes the previously obtained necessary condition (6), imposed on the decay probability ([2–6]); therefore (9) may be regarded as its natural generalization for decaying physical systems whose energy distribution has a finite first-order moment. Let us note that earlier ([4–6]) the restriction (9) was obtained as a necessary condition of finiteness (for finite $\tilde{\omega}(E)$ the first-order moment surely exists) of the energy distribution $\tilde{\omega}(E)$ *. It follows from what was obtained above that the restriction (9), as a necessary condition of finiteness, is too broad.

The convergence of the integral (9) at the upper limit is automatically ensured by the fulfillment of the necessary condition (6) in the quantum theory of decay ([2–6]). Its convergence at the lower limit, however, leads to a new physical consequence, namely, to a restriction on the rate of change of the decay law of a physical system (in particular, it cannot be exponential) at the beginning of the decay, $t \sim 0$. In this connection it is interesting to note that, from the spectrality principle (3) (the semi-finiteness of $\tilde{\omega}(E)$), on the basis of the necessary condition (6), imposed on the modulus of the characteristic function $p_1(t)$ (and consequently also on the decay probability $L(t)$), it follows that the decay law cannot be very fast at the end of the decay, $t \to \infty$, namely, it must be slower than exponential ([2–6]) **.

4. From (5), with the use of (8), there follows the inequality

$$\infty > \tilde{\omega}^{(1)} \geq -\frac{1}{\pi} \int_0^\infty \frac{\ln L(t)}{t^2}\, dt > 0, \tag{10}$$

which gives a lower bound for the first-order moments of the energy distributions (and, consequently, also for the "effective" masses of unstable particles***), corresponding to the given decay law $L(t) = M_1^2(t)$. From (5) it also follows that the minimal value $\tilde{\omega}^{(1)}_{\text{min}}$, corresponding to the given decay law $L(t)$, is determined by the expression

$$\tilde{\omega}^{(1)}_{\text{min}} = -\frac{1}{\pi} \int_0^\infty \frac{\ln L(t)}{t^2}\, dt. \tag{11}$$

As was already noted, the problem of reconstructing the energy distribution $\tilde{\omega}(E)$ from a given decay law $L(t)$ is, generally speaking, not unique ([2–6]), the nonuniqueness being due to possible zeros of $p_1(t)$ in the upper half-plane $\operatorname{Im} t > 0$. On the basis of the result (5) obtained above, it is not difficult to obtain an expression for the first-order moments of the energy distributions $\tilde{\omega}(E)$ corresponding to one and the same decay law $L(t)$. If we denote by $\tilde{\omega}^{(1)}_K$ the first-order moment corresponding to the decay law $L(t)$ and to the presence in $p_1(t)$ of K zeros in the upper half-plane, and by $\tilde{\omega}^{(1)}_0$ the first-order moment corresponding to the same decay law $L(t)$ and to the absence of zeros of $p_1(t)$, then from (5) we obviously obtain:

$$0 < \tilde{\omega}^{(1)}_K = \tilde{\omega}^{(1)}_0 + \sum_{k=1}^{K} \frac{2\, \operatorname{Im} t_k}{|t_k|^2} < \infty, \tag{12}$$

* Finite energy distributions probably describe strongly interacting unstable particles (for example, the π-meson), decaying by means of the weak interaction ([6,8]).

** This result has recently been repeated in works ([7,8,14,15]).

*** Unstable particles do not have a definite value of mass (unlike stable ones), and one can speak only of an "effective" mass.

i.e., the presence of complex zeros of $p_1(t)$ increases the first-order moment of the energy distribution. This result is of interest in connection with the proposal, mentioned earlier, by Matthews and Salam[10] to determine the mass of an unstable particle through the first-order moment of the energy distribution.

From the result obtained above it follows that the condition of absence of complex zeros of $p_1(t)$ in the upper half-plane $\operatorname{Im} t > 0$ (which has no direct physical meaning) is equivalent to the requirement of the minimum value of the first-order moment, corresponding to the given decay law $L(t)$, of the energy distribution $\tilde{\omega}(E)$—a requirement that has direct physical meaning.

5. On the basis of the theorem (5) formulated above, it is not difficult to show that the problem of reconstructing the energy distribution $\tilde{\omega}(E)$ in the class of energy distributions having a finite first-order moment,

$$\tilde{\omega}^{(1)} = -\frac{1}{\pi} \int_0^{\infty} \frac{\ln L(t)}{t^2}\, dt, \tag{13}$$

from a given decay law $L(t)$ is unique. In this case the following expression is obtained for the required energy distribution:

$$\tilde{\omega}(E) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{L(t)} \exp\left\{ -iEt - i\frac{t}{\pi} P \int_0^{\infty} \frac{\ln L(t') - \ln L(t)}{t'^2 - t^2}\, dt' \right\} dt. \tag{14}$$

Indeed, from (13), on the basis of the theorem (5) obtained above, it follows that the energy distributions corresponding to the given decay law are such that the principal function $p_1(t)$ has no zeros in the upper half-plane $\operatorname{Im} t > 0$. And then, on the basis of the analyticity of $p_1(t)$, it follows from the dispersion relations (2-6) that $p_1(t)$ is uniquely determined by its modulus $M_1(t)$, i.e., by the given decay law $L(t) = M_1^2(t)$, since

$$N_1(t) = \frac{t}{\pi} P \int_0^{\infty} \frac{\ln L(t') - \ln L(t)}{t'^2 - t^2}\, dt', \tag{15}$$

and then, on the basis of (2), $\tilde{\omega}(E)$ is uniquely determined by the expression (14), as was required to prove.

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Note: Figure translations are in progress. See original paper for figures.