# MATHEMATICS

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## Abstract

## Full Text

MATHEMATICS

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# ON CONNECTION FORMULAS FOR SINGULAR TRICOMI PROBLEMS

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We shall call singular Tricomi problems the problem of finding, in the domain $D$ $(y > x > 0)$, those solutions $u(x, y, \beta)$ and $\bar{u}(x, y, \beta)$ of the Euler–Poisson equation

$$(y - x)u_{xy} + \beta(u_x - u_y) = 0 \qquad (0 \leq a = 2\beta < 1), \tag{1}$$

which are continuous in $D$ together with their second-order derivatives and satisfy, along the half-lines $y = x \geq 0,\ x = 0,\ y \geq 0$, respectively, the conditions

$$u(x, x) = f(x), \qquad u(0, y) = 0; \tag{2}$$

$$\bar{u}_\eta(x, x) = f(x), \qquad \bar{u}(0, y) = 0; \qquad \eta = -((y - x)/(2 - 2a))^{1-a}. \tag{3}$$

Here $f(x)$ is assumed to be twice continuously differentiable on the semiaxis $y = 0,\ x \geq 0$, with $f(0) = 0$.

**Theorem 1.** If $\beta_2 > \beta_1 \geq 0,\ \beta = \beta_2 - \beta_1,\ a = a_2 - a_1,\ \varkappa_1\Gamma(\beta)\Gamma(1/2 - \beta_2) = 2^a\Gamma(1/2 - \beta_1)$, then the connection formula holds

$$u(x, y, \beta_2) = (y - x)^{1-\beta_1-\beta_2} \int_0^x K_1(x, y, \xi, \beta_1, \beta_2) u(\xi, y, \beta_1)\, d\xi, \tag{4}$$

where

$$K_1 = \varkappa_1(y - \xi)^{a_1 - 1}(x - \xi)^{\beta - 1} F(-\beta, \beta_2, \beta; \omega).$$

**Theorem 2.** Let $\beta_1 > \beta_2 \geq 0,\ \bar{\beta} = \beta_1 - \beta_2,\ \varkappa_2(1 - a_1)^{a_1}\Gamma(\bar{\beta})\Gamma(1/2 - \beta_2) = (1 - a_2)^{a_2}\Gamma(1/2 + \beta_1)$. Then the functions $\bar{u}(x, y, \beta_i)$ $(i = 1, 2)$ are connected by the equality

$$\bar{u}(x,y,\beta_2) = (y-x)^{\bar{\beta}} \int_0^x K_2(x,y,\xi,\beta_1,\beta_2)\bar{u}(\xi,y,\beta_1)\,d\xi, \tag{5}$$

whose kernel is

$$K_2 = \varkappa_2 (x-\xi)^{\bar{\beta}-1} F(-\bar{\beta}, 1-\beta_2, \bar{\beta}; \omega).$$

**Theorem 3.** If $\beta_1 \geq 0,\ \beta_2 \geq 0,\ \beta_1+\beta_2 \neq 0,\ \varkappa_3(1-a_1)^{a_1}\Gamma(1/2-\beta_2) \times \Gamma(\beta_1+\beta_2) = 2^{a_2}\Gamma(1/2+\beta_1)$, then $\bar{u}(x,y,\beta_1)$ is transformed into $u(x,y,\beta_2)$ by the relation

$$u(x,y,\beta_2) = (y-x)^{\bar{\beta}} \int_0^x K_3(x,y,\xi,\beta_1,\beta_2)\bar{u}_\xi(\xi,y,\beta_1)\,d\xi, \tag{6}$$

where

$$K_3 = \varkappa_3 (x-\xi)^{\beta_1+\beta_2-1} F(1-\beta_1-\beta_2, \beta_2, \beta_1+\beta_2; \omega).$$

The inverse connection is given by the solution of the integral equation (6):

$$\bar{u}(x,y,\beta_2) = (y-x)^{\beta_1} \int_0^x K_4(x,y,\xi,\beta_1,\beta_2)u(\xi,y,\beta_1)\,d\xi, \tag{7}$$

valid for $\beta_1 \geq 0,\ \beta_2 \geq 0,\ \beta_1+\beta_2 < 1,\ 2^{a_1}\chi_4\Gamma(1/2+\beta_2)\Gamma(1-\beta_1-\beta_2) = (1-a_2)^{a_2}\Gamma(1/2-\beta_1)$,
$K_4 = \chi_4(x-\xi)^{-\beta_1-\beta_2}(y-\xi)^{-\beta_2}F(2-a_1-a_2,\ -\beta_1,\ 1-\beta_1-\beta_2;\omega)$.

Let us compare the Tricomi problems under consideration with the solutions $z(x,y,\beta)$ and $\bar{z}(x,y,\beta)$ of two singular Goursat problems (1):

$$yz_{xy} + \alpha z_y + \beta z_x = 0 \qquad (\alpha > 0,\ \beta \geq 0,\ (x,y) \in D); \tag{8}$$

$$z(0,y) = 0, \qquad z(x,0) = f(x); \qquad \bar{z}(0,y) = 0, \qquad \bar{z}_y(x,0) = f(x), \tag{8a}$$

assuming, as before, that $f(0) = 0,\ f(x) \subset L_2$.

**Theorem 4.** Assuming $\beta_2 > \beta_1 \geq 0,\ \bar{D}_x = \partial/\partial x,\ y\omega = \alpha(x-\xi),\ \Gamma(1-\beta_2) = \mu_1\Gamma(1-a_2)\Gamma(1+\beta)$, we arrive at the transformation formula:

$$u(x,y,\beta_2) = (y-x)^{-\beta_2}\left(\frac{y}{\alpha}\right)^{\beta_1} e^{-\alpha x/y} \int_0^x Q_1 z(\xi,y,\beta_1)\,d\xi. \tag{9}$$

This time the kernel $Q_1$ reduces to Humbert's confluent hypergeometric function ($^2$):
$Q_1 = -\mu_1 \exp(\alpha\xi/y) D_\xi[(x-\xi)^\beta \Phi_1(\beta_2, 1-\beta_2,\ 1+\beta,\ \omega, \bar{\omega})]$. When $\beta_2 = \beta_1 = \beta$, and $\bar{Q}_1 = Q_1(x, y, \xi, \beta, \beta)$, formula (9) takes the form

$$\left[\frac{\alpha(y-x)}{y}\right]^\beta u(x, y, \beta) = \frac{\Gamma(1-\beta)}{\Gamma(1-\alpha)} z(x, y, \beta) + e^{-\alpha x/y} \int_0^x \bar{Q}_1 z(\xi, y, \beta)\, d\xi.$$

Inverting the relations (9), we find, for $\beta_2 > \beta_1 \geq 0,\ \mu_2\Gamma(1-\beta_1)\Gamma(1+\beta) = \Gamma(1-a_1)$,

$$z(x, y, \beta_2) = \left(\frac{y}{\alpha}\right)^{-\beta_2} (y-x)^{1-\beta_1} \int_0^x Q_2 u(\xi, y, \beta_1)\, d\xi, \tag{10}$$

where

$$Q_2 = -\mu_2 (y-\xi)^{a_1-1} D_\xi[(x-\xi)^\beta \Phi_1(\beta_2, \beta_1-1,\ 1+\beta; \omega, -\bar{\omega})].$$

For equal values $\beta_2 = \beta_1 = \beta$ and $Q_2(x, y, \xi, \beta, \beta) = Q_2$,

$$\left[\frac{\alpha(y-x)}{y}\right]^{-\beta} z(x, y, \beta) = \frac{\Gamma(1-\alpha)}{\Gamma(1-\beta)} u(x, y, \beta) + (y-x)^{1-\alpha} \int_0^x \bar{Q}_2 z(\xi, y, \beta)\, d\xi.$$

**Theorem 5.** Let $\bar{\beta}_1 < 1,\ \mu_3\Gamma(1+\beta_2)\Gamma^2(1-\beta_1) = \alpha\Gamma(1-a_1)$; $\gamma(b, z)$ is Euler' s gamma function ($^3$). Then:

$$\bar{z}(x, y, \beta_2) = \int_0^x Q_3(x, y, \xi, \beta_1, \beta_2) u(\xi, y, \beta_1)\, d\xi, \tag{11}$$

$$Q_3 = \mu_3 (y-\xi)^{a_1-1} D_\xi \int_\xi^x (y-t)^{1-\beta_1} (t-\xi)^{-\beta_1} \gamma\left[\beta_2, \frac{\alpha(x-t)}{y}\right] dt.$$

The functions $\bar{z}(x, y, \beta_2)$, $u(x, y, \beta_1)$ also satisfy an analogous relation. It is also of interest to compare (1) with the equation

$$y v_{yy} + \beta v_y - \alpha v_x = 0 \qquad (\alpha > 0, (x, y) \in D).$$

If for $v(x, y, \beta)$ the first of conditions (8a) are fulfilled, then, denoting by $Q_4(x, y, \xi, \beta_1, \beta_2)$ $(0 \leq \xi \leq x)$ the function

$$Q_4 = -\mu_4 (y-\xi)^{a_1-1} D_\xi \int_\xi^x (x-t)^{\beta_2-2}(y-t)^{1-\beta_1}(t-\xi)^{-\beta_1} \exp\left(-\frac{\alpha y}{x-t}\right) dt,$$

where $\mu_4 \Gamma^2(1-\beta_1)\Gamma(1-\beta_2) = \Gamma(1-a_1)$, we obtain

$$v(x,y,\beta_2) = (\alpha y)^{1-\beta_2} \int_0^x Q_4(x,y,\xi,\beta_1,\beta_2) u(\xi,y,\beta_1)\, d\xi. \tag{12}$$

The formulas relating $v$ to $\bar{u}$, and also $\bar{v}$ to $u(x,y,\beta_1)$, have an analogous form. Along with (1), the more general singular equation (4) was studied earlier:

$$(y-x)w_{xy} + \beta(w_x - w_y) - b^2(y-x)w = 0.$$

For its solutions $w(x,y,\beta)$ satisfying the conditions (2), we obtain, putting $r = \sqrt{(x-t)(y-t)}$, $z^\nu I_\nu(z) = 2^\nu \Gamma(1+\nu) I_\nu(z)$:

$$w(x,y,\beta_2) = (y-x)^{1-a_2} \int_0^x R_1 u(\xi,y,\beta_1)\, d\xi, \tag{13}$$

if $\beta_2 > \beta_1 \geq 0$, $\nu_1 \Gamma(\beta_2)\Gamma(1-\beta_1)\Gamma(1/2-\beta_2) = 2^a \Gamma(1/2-\beta_1)$,

$$R_1 = -\nu_1 (y-\xi)^{a_1-1} D_\xi \int_\xi^x (x-t)^{\beta_2-1}(y-t)^{\bar{\beta}}(t-\xi)^{-\beta_1} \bar{I}_{\beta_2-1}(2br)\, dt.$$

In constructing similar relations for the integrals $\bar{w}(x,y,\beta)$, satisfying the boundary data (3), one must, conversely, take $\beta_1 > \beta_2 \geq 0$, which gives, if

$$\nu_2(1-a_1)^{a_1}\Gamma(1-\beta_2)\Gamma(\beta_1)\Gamma(\beta_2+1/2) = (1-a_2)^{a_2}\Gamma(\beta_1+1/2):$$

$$\bar{w}(x,y,\beta_2) = \int_0^x R_2(x,y,\xi,\beta_1,\beta_2)\bar{u}(\xi,y,\beta_1)\, d\xi,$$

$$R_2 = -\nu_2 D_\xi \int_\xi^x (x-t)^{-\beta_2}(y-t)^{\bar{\beta}}(t-\xi)^{\beta_1-1}\bar{I}_{-\beta_2}(2br)\, dt. \tag{14}$$

At the same time, for an unchanged value of the parameter $\beta$,

$$w(x,y,\beta) = u(x,y,\beta) + (y-x)^{1-a} \int_0^x R_1(x,y,\xi,\beta,\beta) u(\xi,y,\beta)\, d\xi$$

$$\bar{w}(x,y,\beta) = \bar{u}(x,y,\beta) + \int_0^x R_2(x,y,\xi,\beta,\beta)\bar{u}(\xi,y,\beta)\, d\xi.$$

As a result of replacing $u$ and $\bar{u}$ by the functions $z$ and $\bar{z}$, these formulas assume the form

$$w(x,y,\beta_2) = \left(\frac{y}{\alpha}\right)^{\beta_1} (y-x)^{1-a_2} \int_0^x R_3 D_\xi \left[e^{\alpha\xi/y} z(\xi,y,\beta_1)\right] d\xi, \tag{15a}$$

$$\bar{w}(x,y,\beta_2) = \left(\frac{y}{\alpha}\right)^{\beta_1} \int_0^x R_4 D_\xi \left[e^{\alpha\xi/y} \bar{z}_\xi(\xi,y,\beta_1)\right] d\xi, \tag{15b}$$

where for $R_3$ we obtain the integral representation

$$R_3 = \nu_3 \int_\xi^x (t-\xi)^{-\beta_1} r^{a_2-2} e^{-\alpha t/y} \bar{I}_{\beta_2-1}(2br)\, dt,$$

in which

$$\nu_3 \Gamma(\beta_2)\Gamma(1-a_2)\Gamma(1-\beta_1) = \Gamma(1-\beta_2),$$

while $R_4$ is determined from the equality

$$\alpha \cdot 2^{1-a_2}\Gamma(a_2) R_4 = -\beta_1(1-a_2)^{a_2}\Gamma(a_2-1) R_3(x,y,\xi,\beta_1,1-\beta_2).$$

After integration by parts, (15) generates four relations corresponding to the conditions $\beta_1 > \beta_2 \geq 0$, $\beta_1+\beta_2 > 1$, and $\beta_1 = \beta_2 = \beta$. Equalities transforming $u$ and $z$ into $\bar{w}$, and also $u$ and $\bar{z}$ into $w$, have an analogous form.

If $f(x) \in L_\infty$, then, by analogy with (1), for the indicated transformation operators one can construct expansions into infinite series, converging absolutely and uniformly in the domain $D$. Namely, we introduce the notation

$$U_x^\beta = k_1 \rho^\beta \Phi_1(\beta, 1-\beta, 1+\beta; -\rho, -\delta_x), \quad \bar{U}_x^\beta = k_2 x^{1-a} \rho^\beta \Phi_1(1-\beta, \beta, 2-\beta; -\rho, -\delta_x),$$

where

$$\rho(y-x) = x, \qquad \delta_x = x D_x,$$

$$k_1 \Gamma(1+\beta)\Gamma(1-a) = \Gamma(1-\beta), \qquad k_2\, 2^{1-a}\Gamma(2-\beta)\Gamma(a) = (1-a)^a \Gamma(\beta).$$

Moreover, consider the inverse operators $(U_x^\beta)^{-1} = k_3(y-x)^{1-\beta} \times D_x[x^{1-\beta}(y-x)^{a-1}\Phi_1(1-\beta, 1-a, 2-\beta, -\beta, -\delta_x)]$, $(\overline{U}_x^\beta)^{-1} = k_4(y-x)^\beta \times D_x[x^\beta \Psi(\beta,\delta_x)]$, $k_3(1-\beta)\Gamma^2(1-\beta) = \Gamma(1-a)$, $\sqrt{\pi}(1-a)^a k_1 = \Gamma(\frac{1}{2}+\beta)$. Then, using the notation of paper (1), we obtain $u_2 = U_x^{\beta_2}(U_x^{\beta_1})^{-1}u_1$; $\overline{u}_2 = \overline{U}_x^{\beta_2}(\overline{U}_x^{\beta_1})^{-1}u_1$; $u_2 = U_x^{\beta_2} Z_x^{1-\beta_1}\Delta z_1$; $u_2 = U_x^{\beta_2}\Delta_x^{\beta_1} z_1$; $z_2 = Z_x^{\beta_2}(U_x^{\beta_1})^{-1}u_1$; $z_2 = \Delta_x^{-\beta_2}(U_x^{\beta_1})^{-1}u_1$; $v_2 = V_x^{\beta_2}(U_x^{\beta_1})^{-1}u_1$; $\overline{v}_2 = \overline{V}_x^{\beta_2}(\overline{U}_x^{\beta_1})^{-1}\overline{u}_1$. Note that $u(x,y,0) = u_x(x,y,0) = z(x,y,0) = f(x)$. Thus, the above equalities for $\beta_1 = 0$ or $\beta_2 = 0$ give explicit formulas for the solution of the corresponding boundary-value problems, or their inversion with respect to the initial functions.

Since $K_1 \geq 0$, it follows from (4) that if $f'(x) > 0$, then, as the parameter $\beta$ increases, $u(x,\bar{y},\beta)$ decreases. Above, the integrals of two equations with an unchanged initial function were related. Let now $f_2(x) = P(x)f_1(x)$, where $P(x)$ is an arbitrary function integrable on every finite interval of the semiaxis $y = 0$, $x \geq 0$. In this case, for the corresponding solutions formulas (11)–(15) also hold, but the integrands of the kernels $Q_3, Q_4, R_i$ $(i = 1, 2, 3, 4)$ acquire the additional factor $P(t)$. Examples of such relations can also be constructed by writing $u, z, v, w$ in the form of Duhamel integrals containing discontinuous solutions of the same problems for $f_1(x) = 1$. Without presenting other results arising from pairwise combinations of the solutions $u_i, \overline{u}_i, v_i, \overline{v}_i, w_i, \overline{w}_i, z_i, \overline{z}_i$ $(i = 1, 2)$ under the condition $f_2(x) = P(x)f_1(x)$, we note only that if $P(x) = (x - x_1)^{-k_1} \dots (x - x_n)^{-k_n}$, the kernels $Q_5$ and $R_i$ $(i = 1, \dots, 4)$ are represented in the form of infinite series in the hypergeometric functions of Lauricella ($^2$). When $b = 0$, such series terminate at their first term, and therefore, for example, $\Gamma(1+\beta)R_1 = \Gamma(\beta_2)\Gamma(1-\beta_1)v_1(y-\xi)^{a_1-1}(y-x)^{\beta}P(x)\times D_{\xi}[(x-\xi)^{\beta}F_D(\beta_2, \beta, k_1, \dots, k_n, 1+\beta, \omega, X_1, \dots, X_n)]$, $(x-x_i)X_i = x-\xi$.

On the other hand, for $P(x) = \exp(kx)$, $b = 0$, $R_1$ is expressed through the confluent hypergeometric function $\Phi_1$. Similar transformation formulas can in turn be generalized to the case when two or more initial functions $f_i(x)$ are connected by an arbitrary, prescribed-in-advance relation. A number of other generalizations are provided by products of the transformation operators found. Thus, for example, in order to solve the integral equation (12) with respect to $u(x, y, \beta_1)$, it suffices in formula (9) to replace $z(x, y, \beta)$ by expression (10) from paper ($^1$), which transforms $z(x, y, \beta)$ into $v(x, y, \beta)$. Substituting the same expression in (15), we connect $w(x, y, \beta)$ with $v(x, y, \beta)$.

With the aid of the connection formulas, for a suitable choice of the values $f_i(x)$, it is also possible to compute a number of integrals with special functions. For example, if $f_1(x) = x^m(1 - cx)^{-k}$, $f_2(x) = x^m e^{kx}$ $(m > 0,\ k < 1)$, then in the corresponding relations there appear the expressions* $u_1 = MF_1(1+m, 1-\beta, k, 1+m+\beta, x/y, cx)$, $u_2 = M\Phi_1(1+m, 1-\beta, 1+m+\beta, x/y, kx)$, where $M\Gamma(1-a)\Gamma(1+m+\beta) = \Gamma(1-\beta)\Gamma(1+m)x^{m+\beta}y^{\beta-1}(y-x)^{1-a}$. In particular, when $k = 0$, $u_1$ and $u_2$ reduce to the hypergeometric functions of Gauss, and for $f(x) = x^m(x_1 - x)^{-k_1} \dots (x_n - x)^{-k_n}$ the solutions $u(x, y, \beta), \overline{u}(x, y, \beta)$ are expressed through the Lauricella functions $F_D$.

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* Analogous values for $z$ and $v$ are indicated in ($^4$).

*Note: Figure translations are in progress. See original paper for figures.*