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Abstract

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ON ALMOST PERIODIC OSCILLATIONS OF QUASI-LINEAR SYSTEMS WITH TIME LAG IN THE DEGENERATE CASE

(Presented by Academician N. N. Bogolyubov on 9 III 1960)

In the works of N. N. Bogolyubov, I. G. Malkin, and T. I. Biryuk, general propositions were obtained on the existence of almost periodic oscillations for nonlinear systems whose motion is described by ordinary nonlinear differential equations with a small parameter ([1–4]). In the present note a proposition is established on the existence of almost periodic oscillations in nonlinear systems with time lag in the degenerate case, when among the roots of the characteristic equation there are so-called critical roots—purely imaginary and zero. The nondegenerate case was considered in ([6]), and the degenerate case for periodic perturbations in ([7]).

Consider a system whose motion is described by the equations

$$\frac{dx(t)}{dt} = \int_{-\tau}^{0} x(t+\vartheta)\, d\eta(\vartheta) + \varepsilon F(t, x(t+\vartheta), \varepsilon), \tag{1}$$

where $x(t)$ is an n-dimensional vector; $d\eta(\vartheta)$ is an n-dimensional matrix; $\|d\eta_{ij}(\vartheta)\|$ is a Stieltjes measure, allowing one to obtain from system (1), in particular, various systems with time lag under particular assumptions on the functions of bounded variation $\eta_{ij}(\vartheta)$.

The characteristic equation

$$\Delta(\lambda) \equiv \left| +E\lambda + \int_{0}^{-\tau} e^{\lambda\vartheta}\, d\eta(\vartheta) \right| = 0 \tag{2}$$

has m critical roots $\lambda_1, \dots, \lambda_m$ and the remaining roots $\lambda_{m+1}, \dots$ with negative real parts less than -2α ($\alpha > 0$).

The functionals $F(t, x(t+\vartheta), \varepsilon) = \{F_i\}$ are defined on piecewise-continuous functions $x(\vartheta)$ ($-\tau \leq \vartheta \leq 0$) in the domain $D(H) : \|x(\vartheta)\| \leq H$ and satisfy the following conditions:

1) In the domain $D_\varepsilon : |\varepsilon| \leq \varepsilon_0$ ($\varepsilon_0 > 0$),

$$F(t, x(\vartheta), \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k F_k(t, x(\vartheta)).$$

2) $F(t, x(\vartheta))$ are finite trigonometric sums if in them one substitutes any function $x_t(\vartheta) \in D(H)$, piecewise continuous in ϑ, which with respect to t is a finite trigonometric sum with frequencies not depending on ϑ.

3) The functionals $F(t, x(\vartheta), \varepsilon)$ satisfy Lipschitz conditions with respect to the variables x in the domain $D(H) \cdot D_\varepsilon$ of the form

$$|F_i(t, x_1''(\vartheta), \dots, x_n''(\vartheta), \varepsilon) - F_i(t, x_1'(\vartheta), \dots, x_n'(\vartheta), \varepsilon)| \le L\|x'' - x'\| \tag{3}$$

$$(L = \text{const}, \quad i = 1, 2, \dots, n).$$

Suppose that to each critical root λ_j $(j = 1, \dots, m)$ there corresponds a periodic solution $\varphi_j(t)$ of the linear system (1) for $\varepsilon = 0$. Then the "adjoint" system

$$\frac{dy}{dt} \int_{\tau}^{0} y(t + \vartheta)\, d\eta^*(\vartheta) \quad (d\eta^* = -\|d\eta_{ji}(-\vartheta)\|) \tag{4}$$

will also have m periodic solutions $\psi_j(t)$, corresponding to the m critical roots $-\lambda_j$ $(j = 1, \dots, m)$.

Theorem 1. Let $x^0(t) = M_1\varphi_1(t) + \dots + M_m\varphi_m(t)$ be an almost periodic solution of system (1) for $\varepsilon = 0$. Then, if the parameters $M_i = M_i^0$ satisfy the equations

$$P_i(M_1, \dots, M_m) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{j=1}^{n} F_j(t, x_1^0(t+\vartheta), \dots, x_n^0(t+\vartheta))\psi_i(t)\, dt \tag{5}$$

$$(i = 1, \dots, m)$$

and the equation

$$\left| \left(\frac{\partial P_i(M)}{\partial M_j} \right)_{M=M^0} - d_{ij}\lambda \right| = 0 \tag{6}$$

has no roots with zero real parts, then, for sufficiently small ε, system (1) admits an almost periodic solution $x^*(t, \varepsilon)$, which for $\varepsilon = 0$ passes into the generating solution $x^0(t, M^0)$ (the d_{ij} are defined below).

Theorem 2. If all noncritical roots of equation (2) have negative real parts and all roots of equation (6) also have negative real parts, then the almost periodic motion $x^*(t,\varepsilon)$ of system (1), which for $\varepsilon = 0$ passes into the generating solution $x^0(t, M^0)$, will be asymptotically stable for sufficiently small ε. If at least one real part of the indicated roots is positive, then the almost periodic motion $x^*(t,\varepsilon)$ under consideration is unstable.

Let us indicate the course of the proof. As an element of a solution of system (1), we shall consider segments of trajectories of the motion of system (1) on the time interval $t+\vartheta$ $(-\tau \leq \vartheta \leq 0)$. In the functional space of trajectory segments, the motion of the system is determined by the vector-functions of time $x_t(\vartheta) = x(t+\vartheta)$. In the functional space $x(\vartheta)$, the linear system (1) for $\varepsilon = 0$ corresponds to a system of "ordinary" differential equations with an operator right-hand side ([5]):

$$\frac{dx_t(\vartheta)}{dt} = Ax_t(\vartheta), \qquad A(\varphi(\vartheta)) = \begin{cases} \dfrac{d\varphi_k}{d\vartheta}, & (-\tau \leq \vartheta < 0), \\ \displaystyle\int_{-\tau}^{0} \varphi(\vartheta)\, d\eta(\vartheta), & (\vartheta = 0). \end{cases} \tag{7}$$

The operator A is defined on any differentiable vector-function $\{\varphi_k(\vartheta)\} = \varphi(\vartheta)$.

Define the adjoint system:

$$\frac{dy_t(\vartheta)}{dt} = A^* y_t(\vartheta), \qquad A^*(\varphi(\vartheta)) = \begin{cases} \dfrac{d\varphi_k}{d\vartheta}, & (0 < \vartheta \leq \tau), \\ \displaystyle\int_{\tau}^{0} \varphi(\vartheta)\, d\eta^*(\vartheta), & (\vartheta = 0). \end{cases} \tag{8}$$

System (8) will be with an advance of time. Define the scalar product of the vector-function $x(\vartheta)$ $(-\tau \leqslant \vartheta \leqslant 0)$ with the vector-function $y(\vartheta)$ $(0 \leqslant \vartheta \leqslant \tau)$:

$$(x \cdot y) \equiv \sum_{j=1}^{n} x_{jt}(0) y_{jt}(0) - \sum_{j=1,\, l=1}^{n} \int_{-\tau}^{0} \left[\int_{0}^{\vartheta} x_{lt}(\xi) y_j(-\vartheta+\xi)\, d\xi \right] d\eta_{jl}(\vartheta). \tag{9}$$

For any two solutions $x_t(\vartheta)$ of system (7) and $y_t(\vartheta)$ of system (8), the condition

$$(x_t(\vartheta) y_t(\vartheta)) = \mathrm{const}$$

holds.

Let $d_{ij} = (\varphi_i(t+\vartheta)\psi_j(t+\vartheta))$. Under the assumptions made about the roots λ_j $(j = 1, \dots, m)$ of the characteristic equation, $|d_{ij}| \neq 0$.

We split the space $x(\vartheta)$ into an m-dimensional subspace $l : \{y_j\}$ and an $\mathcal{L}$-functional subspace with index m, setting

$$x_t(\vartheta) = z_t(\vartheta) + \varphi_1(t+\vartheta)y_1 + \dots + \varphi_m(t+\vartheta)y_m; \tag{10}$$

$$f_j[z_t(\vartheta)] = 0 \qquad (j = 1, \dots, m), \tag{11}$$

where $f_j[z_t(\vartheta)] = (z_t(\vartheta)\psi_j(t+\vartheta))$.

In this case the variables $y_1, \dots, y_m$ are uniquely determined from the system of equations

$$f_j[x_t(\vartheta)] = d_{j1}y_1 + \dots + d_{jm}y_m. \tag{12}$$

Since l and $\mathcal{L}$ have no common elements, except for the point $x(\vartheta) = 0$, (11) and (12) ensure the uniqueness of the representation (10).

It can be shown that, for every function $x_0(\vartheta)$ satisfying conditions (11), the corresponding motion $x_t(\vartheta)$ of system (7) will satisfy conditions (11) for all $t \geqslant 0$ and will decrease asymptotically according to an exponential law with exponent $-\alpha$.

The complete system (1) in the space $x_t(\vartheta)$ has the form

$$\frac{dx_t(\vartheta)}{dt} = Ax_t(\vartheta) + \varepsilon R(t, x_t(\vartheta), \varepsilon), \tag{13}$$

where

$$R = \begin{cases} 0, & -\tau \leqslant \vartheta < 0, \\ \varepsilon F, & \vartheta = 0. \end{cases}$$

In the new variables $y_1, \dots, y_m, z_t(\vartheta)$ it will have the form

$$\frac{dy_i}{dt} = \varepsilon \sum_{j=1}^{m} \frac{\Delta_{ij}}{\Delta} f_j[R(t, x_t(\vartheta), \varepsilon)]; \tag{14}$$

$$\frac{dz_t(\vartheta)}{dt} = Az_t(\vartheta) + \varepsilon R(t, x_t(\vartheta), \varepsilon) - \varepsilon \sum_{i=1}^{m} \varphi_i(t+\vartheta) \sum_{j=1}^{n} \frac{\Delta_{ij}}{\Delta} f_j[R],$$

where $\Delta = |d_{ij}|$; Δ_{ij} is the algebraic cofactor of the element d_{ij} located at the intersection of the i-th row and the j-th column; in the right-hand sides of (14), $x_t(\vartheta)$ should be taken as defined by formulas (10)–(12). For $\varepsilon = 0$, system (14) admits an m-parameter family of almost periodic motions $y_i^0 = M_i$, $z_t^0(\vartheta) = 0$. Next, system (14) can be subjected to the transformation of N. N. Bogoliubov and N. M. Krylov. Applying to the system thus obtained the method of successive approximations, it is not difficult to prove Theorem 1. To prove Theorem 2 we apply

the method of Lyapunov functions, replacing them by the corresponding functionals and using scheme (3), p. 267.

An almost periodic solution of the system

$$\frac{dx_t(\vartheta)}{dt} + Ax_t(\vartheta) + F(t,\vartheta) \tag{15}$$

(where $F(t,\vartheta)$ is an almost periodic function of t and a piecewise-continuous function of ϑ with discontinuities of the first kind) is determined by Cauchy' s formula

$$x^*(t,\vartheta) = \int_0^t T(t-t_1)F(t_1,\vartheta)\,dt_1,$$

provided the condition $f_j[F(t,\vartheta)] = 0$ $(j = 1,\dots,m)$ is fulfilled. Here $T(t)$ is the semigroup operator determining the solution $x_t(\vartheta) = T(t)x_0(\vartheta)$ of the homogeneous linear system (15). If $x_0(\vartheta) \in \mathcal{L}$, then $T(t)x_0(\vartheta)$ decreases according to an exponential law, and $T(0)x_0(\vartheta) = x_0(\vartheta)$.

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Note: Figure translations are in progress. See original paper for figures.