Mathematics

1960

SovietRxiv

Abstract

Full Text

Mathematics

K. I. BABENKO

ON THE APPROXIMATION OF A CLASS OF PERIODIC FUNCTIONS OF MANY VARIABLES BY TRIGONOMETRIC POLYNOMIALS

(Presented by Academician M. V. Keldysh on 31 XII 1959)

§ 1. In the note ([1]) we formulated certain problems in the theory of approximation of periodic functions of many variables. Below we shall use the notation of the note ([1]). Here we shall confine ourselves to considering one remarkable special case.

Consider the collection of periodic functions $f(x)$ subject to the condition

$$\sup_{x\in K}\left|\frac{\partial^{l_1+\cdots+l_m}f(x)}{\partial x_1^{l_1}\cdots\partial x_m^{l_m}}\right|\leq 1, \tag{1}$$

where $l_1, l_2, \ldots, l_m$ are integers.

The case of fractional derivatives is studied analogously and entails no changes in the details of the proof. Let $0<\lambda_1<\lambda_2<\cdots$ be the eigenvalues and $\varphi_1,\varphi_2,\ldots$ the eigenfunctions of the operator

$$P(D)=(-1)^{l_1+l_2+\cdots+l_m}p_1^{2l_1}\cdots p_m^{2l_m}. \tag{2}$$

The problem of approximating functions of the class F_P naturally leads to the consideration of the zeta-function of the operator (2). Put

$$\xi(s,x;P)=\sum_1^\infty\frac{\varphi_n(x)}{\lambda_n^s}.$$

If $0<x_j<1,\ j=1,2,\ldots,m$, then $\xi(s,x;P)$ is an entire function of the variable s. For the operator (2) the zeta-function will satisfy a functional equation which is a consequence of Hurwitz' s functional equation.

Indeed, put

$$\xi(s,x;l)=\sum_{n=1}^{\infty}\frac{\cos(2\pi nx-\pi l/2)}{n^l s};$$

then

$$\xi(s,x;P)=\prod_{j=1}^{m}\xi(2s,x_j;l_j).$$

For the function $\xi(s,x;l)$ the functional equation is known:

$$\xi(s,x;l)=2^{s-1}\pi^{s-1}\Gamma(1-s)\sin\frac{\pi(s+l)}{2}\left[\zeta(1-s\mid x)+(-1)^l\zeta(1-s\mid 1-x)\right],$$

where

$$\zeta(s\mid x)=\sum_{n=0}^{\infty}\frac{1}{(n+x)^s},\quad 0<x<1.$$

Therefore

$$\zeta\left(\frac{s}{2},x;P\right)=\prod_{j=1}^{m}2^{sl_j-1}\pi^{sl_j-1}\Gamma(1-sl_j)\sin\frac{\pi l_j(s+1)}{2}[\zeta(1-sl_j\mid x_j)+(-1)^{l_j}\zeta(1-sl_j\mid 1-x_j)]. \tag{3}$$

We now proceed to the construction of approximating polynomials. Any function in F_P can be represented in the form:

$$f(x)=\int_K h(y)\zeta\left(\frac{1}{2},x-y,P\right)dy+g(x), \tag{4}$$

where $g(x)\in\mathcal{L}$, $\quad h(x)=\partial^{l_1+\cdots+l_m}f/\partial x_1^{l_1}\cdots\partial x_m^{l_m}$.

Let $Q(x)$ be some polynomial. Then

$$f(x)-g(x)-Q(x)=\int_K h(y)\left[\zeta\left(\frac{1}{2},x-y;P\right)-\widetilde{Q}(x-y)\right]dy,$$

where $\widetilde{Q}$ is a polynomial. Hence

$$|f-g-Q|\le\int_K\left|\zeta\left(\frac{1}{2},x;P\right)-\widetilde{Q}(x)\right|dx. \tag{5}$$

The problem of finding those $\widetilde{Q}$ which minimize the integral on the right-hand side of formula (5) is very difficult. We shall indicate those polynomials $\widetilde{Q}$ for which this integral will have the asymptotically correct order.

Take $\widetilde{Q}$ in the form

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\zeta\left(\frac{s+1}{2},x;P\right)\frac{\theta(s)}{s}N^s\,ds,\qquad \alpha>0,$$

where $\theta(s)$ is defined as follows.

Let $\psi(x)$ be a function satisfying the conditions: 1) $\psi(x)=1$ for $0\le x\le 1/2$; 2) $\psi^{(\nu)}(1/2)=\psi^{(\nu)}(1)=0$ for $\nu=1,2,\dots,$ $\psi(0)=1$.

Denote by $\theta(s)/s$ the Mellin transform of the function $\psi(x)$. Note that $\theta(s)$ is an entire function and $\theta(0)=1$. It is clear that

$$\widetilde{Q}(x)=\sum_{\lambda_{\bar k}^{l}<N^2}\psi\left(\frac{\sqrt{\lambda_{\bar k}}}{N}\right)\frac{\varphi_k(x)}{\sqrt{\lambda_{\bar k}}}=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\zeta\left(\frac{s+1}{2},x;P\right)\frac{\theta(s)}{s}N^s\,ds.$$

Shifting the contour of integration to the left, we obtain

$$\widetilde{Q}(x)=\zeta\left(\frac{1}{2},x;P\right)+\frac{1}{2\pi i}\int_{-\beta-i\infty}^{-\beta+i\infty}\zeta\left(\frac{s+1}{2},x;P\right)\frac{\theta(s)}{s}N^s\,ds,\qquad \beta>0.$$

Denote the last integral by I. Applying relation (3), we obtain

$$I=\frac{N^{-1}}{2\pi i}\int_{-\beta-i\infty}^{-\beta+i\infty}\prod_{j=1}^{m}(2\pi)^{sl_j-1}\Gamma(1-sl_j)\sin\frac{\pi l_j(s+1)}{2}\left[\zeta(1-sl_j\mid x_j)+(-1)^{l_j}\zeta(1-sl_j\mid 1-x_j)\right]\frac{\theta(s-1)}{s-1}N^s\,ds.$$

Suppose that $0<x_j\le 1/2,\ j=1,2,\dots,m$. Then

$$\zeta(1-sl_j\mid x_j)+(-1)^{l_j}\zeta(1-sl_j\mid 1-x_j)=x_j^{sl_j-1}+\xi(s,x_j),$$

where $\xi_j(s,x_j)$ are analytic and bounded functions in the half-plane $\operatorname{Re}s<0$. The principal term of the integral I will be

$$I_1=\frac{N^{-1}}{2\pi i x_1\dots x_m}\int_K\prod_{j=1}^{m}(2\pi)^{sl_j-1}\Gamma(1-sl_j)\sin\frac{\pi l_j(s+1)}{2}\,\frac{\theta(s-1)}{s-1}\left(Nx_1^{l_1}\dots x_m^{l_m}\right)^s\,ds=$$

$$= \frac{F\left(N x_1^{l_1} \ldots x_m^{l_m}\right)}{x_1 \ldots x_m}.$$

It is easy to see that

$$\int_{0<x_j\le 1/2} |I_1|\, dx \le \frac{1}{l_1 l_2 \ldots l_m (m-1)!} \int_0^N \frac{|F(t)|}{t} \log^{m-1} \frac{N}{t}\, dt.$$

Therefore

$$\int_0^1 \left|\zeta\left(\frac{1}{2}, x; P\right) - \widetilde{Q}(x)\right| dx \le C \frac{\log^{m-1} N}{N}. \tag{6}$$

The appearance of $\log^{m-1} N$ in the estimate of the last integral is due to the fact that we considered only those polynomials $\widetilde{Q}(x)$ which have the form

$$\sum_{\lambda_k \le N^2} \psi\left(\frac{\sqrt{\lambda_k}}{N}\right) \varphi_k(x).$$

For these polynomials, the coefficients of the eigenfunctions corresponding to equal eigenvalues are also equal. The polynomial which minimizes the integral (5) does not satisfy this restriction, and the minimum of the integral will be $O(1/N)$. Let us define the number of exponents in the polynomial $Q(x)$, and denote this number by $M(N)$. Then

$$M(N) = \frac{2^m}{2\pi i} \int_{1+\varepsilon-i\infty}^{1+\varepsilon+i\infty} \prod_{j=1}^{m} \zeta(l_{js}) \frac{N^s}{s}\, ds. \tag{7}$$

If $l = \min(l_1, l_2, \ldots, l_m)$ and ν is the number of those l_j which are equal to l, then

$$M(N) = C_\nu 2^m N^{1/l} \log^{\nu-1} N + O\left(N^{1/l} \log^{\nu-2} N\right),$$

where C_ν is a constant depending only on ν.

Thus, if $f \in \mathcal{L}_P$, then there always exists a polynomial $Q(x)$ containing no more than M distinct exponents such that

$$|f(x) - h(x) - Q(x)| \le C \frac{\log^{m-1+l(\nu-1)} M}{M^l}. \tag{8}$$

Let us now consider the class of functions which, in K, satisfy the conditions

$$\left|\frac{\partial^{l_{j_1}+l_{j_2}+\dots+l_{j_s}} f(x)}{\partial x_{j_1}^{l_{j_1}} \dots \partial x_{j_m}^{l_{j_m}}}\right| \leq 1, \tag{9}$$

where $1 \leq j_1 < j_2 < \dots < j_s \leq m,\ 1 \leq s \leq m$.

Applying successively inequality (8), we find a polynomial $\Pi(x)$, containing no more than M distinct exponentials, for which

$$|f(x) - \Pi(x)| \leq C_1 \frac{\log^{m-1+l(\nu-1)} M}{M^l}.$$

We note that the polynomial $\Pi(x)$ contains precisely those exponentials $e^{2\pi i n x}$ for which the vector $n = (n_1, \dots, n_m)$ satisfies the condition

$$\bar{n}_1^{l_1} \cdots \bar{n}_m^{l_m} \leq N, \tag{10}$$

where $\bar{n}_j = |n_j|$ for $n_j \neq 0$ and $\bar{n}_j = 1$ for $n_j = 0$, while N is the root of the equation

$$M(N) = N.$$

The function $M(N)$ is defined by relation (7).

§ 2. The approximation theorem obtained can be applied to the problem of constructing quadrature formulas. If we wish to construct a quadrature formula giving good accuracy for functions satisfying conditions (9), we must choose the nodes of the quadrature formula so that the quadrature formula is exact for all polynomials whose frequency spectrum satisfies condition (10).

The error of the quadrature formula will then be

$$\int_K f(x)\, dx - \frac{1}{M} \sum_1^M f(\xi_k) =$$

$$= \int_K \Pi\, dx - \frac{1}{M} \sum_{k=1}^M \Pi(\xi_k) + O\left(\frac{\log^{m-1+l(\nu-1)} M}{M^l}\right) = O\left(\frac{\log^{m-1+l(\nu-1)} M}{M^l}\right),$$

where ξ_j are the nodes of the quadrature formula.

The problem of constructing nodes satisfying the stated requirement, for $m > 1$, is by no means as trivial as for $m = 1$. A certain type of nodes is indicated in the work of N. M. Korobov ([2]). In that work it is shown that the construction of the nodes of a quadrature formula is connected with delicate number-theoretic questions.

§ 3. Let us dwell once more on the question of inequalities for derivatives of a trigonometric polynomial. Suppose that the spectrum of the trigonometric polynomial $\Pi(x)$ satisfies condition (10) and

$$|\Pi(x)| \leq 1.$$

Then, by the previous method, using the zeta-function and a contour integral, one can show that

$$\left|\frac{\partial^{l_1+\cdots+l_m}\Pi(x)}{\partial x_1^{l_1}\cdots\partial x_m^{l_m}}\right| \leq CN\log^{m-1} N,$$

where C depends only on m.

The factor $\log^{m-1} N$ appearing on the right-hand side of the last inequality arose because of the imperfection of the method of proof. We were unable to obtain a more precise estimate.

V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

Received
31 XII 1959

CITED LITERATURE

1. K. M. Babenko, DAN, **132**, No. 2 (1960). N. M. Korobov, DAN, **124**, No. 6 (1959).
2. DAN, vol. 132, No. 5.

Note: Figure translations are in progress. See original paper for figures.