



Soviet-era science, translated into English

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1960

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Abstract

Full Text

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ON THE SOLUTION OF EQUATIONS OF RIEMANN TYPE IN THE CLASS OF DIRICHLET SERIES

(Presented by Academician P. S. Aleksandrov, 11 IX 1959)

§ 1. Introduction.

Hamburger proved that every ordinary Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

satisfying the Riemann functional equation

$$f(s)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right) = f(1-s)\pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right) \quad (1)$$

and certain additional assumptions concerning the regularity of $f(s)$, has the form $f(s) = c\zeta(s)$, where $\zeta(s)$ is the Riemann function and c is a constant. Hamburger also gave some generalizations of his theorem to the case of the functional equation for L -series. Bochner, Chandrasekhar, and Mandelbrojt^(1,2) considered the equation of a somewhat more general form

$$f(s)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right) = g(\delta-s)\pi^{-(\delta-s)/2}\Gamma\left(\frac{\delta-s}{2}\right) \quad (2)$$

and showed that, for two given sequences $\{\lambda_n\}$ and $\{\mu_n\}$,

$$0 < \lambda_1 < \lambda_2 < \dots, \quad \lambda_n \rightarrow \infty;$$

$$0 < \mu_1 < \mu_2 < \dots, \quad \mu_n \rightarrow \infty,$$

one of which has finite upper density, and for fixed $\delta > 0$, equation (2) has a finite number of linearly independent solutions of the form

$$\left\{ f(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}, \quad g(s) = \sum_{n=1}^{\infty} \frac{b_n}{\mu_n^s} \right\}.$$

Moreover, under additional assumptions concerning $\{\lambda_n\}$ or $\{\mu_n\}$, these authors proved several theorems on the form of the solutions.

In the present paper we consider the class of equations of the form

$$f(s)P(s)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right) = g(1-s)Q(s)\pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right), \quad (3)$$

where $\Gamma(s)$ is Euler's function; $P(s)$ and $Q(s)$ are arbitrary fixed polynomials without common divisors, $P(s)$ is of degree m , and $Q(s)$ is of degree l .

Definition. Let two sequences $\{\lambda_n\}$ and $\{\mu_n\}$ be given,

$$0 < \lambda_1 < \lambda_2 < \dots, \quad \lambda_n \rightarrow \infty; \quad 0 < \mu_1 < \mu_2 < \dots, \quad \mu_n \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

We shall call a pair of functions $\{f(s), g(s)\}$ a **solution of equation (3), belonging to the label $\{\lambda_n, \mu_n\}$** , if:

- 1) There exists a sequence of complex numbers $\{a_n\}$, among which at least one $a_n \neq 0$, and a sequence $\{b_n\}$, for which

at least one $b_n \neq 0$, such that the functions $f(s)$ and $g(s)$ are representable in the form of general Dirichlet series:

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}, \quad g(s) = \sum_{n=1}^{\infty} \frac{b_n}{\mu_n^s},$$

where both series have abscissas of convergence σ_f and σ_g :

$$-\infty \leq \sigma_f < \infty, \quad -\infty \leq \sigma_g < \infty \quad (s = \sigma + i\tau).$$

- 2) There exists a function $\chi(s)$, analytic and single-valued in the domain $|s| > R$ (where R is some constant), tending to zero as $\tau \rightarrow \infty$ uniformly in any bounded strip $\sigma_1 \leq \sigma \leq \sigma_2$, and, for sufficiently large σ , the function $\chi(s)$ coincides with the right-hand side of equation (3), while for sufficiently small σ it coincides with the left-hand side of equation (3), i.e.

$$\chi(s) = f(s)P(s)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right) \quad \text{for } \sigma > \alpha,$$

$$\chi(s) = g(1-s)Q(s)\pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right) \quad \text{for } \sigma < \beta.$$

§ 2. Main theorems on solutions of equation (3).

Theorem 1. Equation (3) has a solution in the class of ordinary Dirichlet series

$$\left\{ f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad g(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s} \right\}$$

in the following and only in the following cases:

1) $P(s) = s(s+1)(s+2) \dots (s+2k-1)$; $Q(s) = \text{const}$.

2) $P(s) = \text{const}$; $Q(s) = (s-1)(s-2) \dots (s-2k)$ (k is a positive integer).

Moreover, in case 1) all solutions of equation (3) have the form $f(s) = c \zeta(s+2k)$, while in case 2) all solutions of (3) have the form $f(s) = c \zeta(s-2k)$.

Theorem 2. If a solution

$$\left\{ f(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}, \quad g(s) = \sum_{n=1}^{\infty} \frac{b_n}{\mu_n^s} \right\}$$

of equation (3) has the property that $\mu_n = m^{(n)} \mu$, where $m^{(n)}$ are integers, then this solution has the form

$$f(s) = \alpha^s \sum_{\lambda < 1} a_{\lambda} [\zeta(s+k, \lambda) + (-1)^k \zeta(s+k, 1-\lambda)] \quad (\alpha > 0),$$

where k is an integer, and λ runs through a finite set of values. $\zeta(s, 0)$ is, by definition, considered equal to $\zeta(s)$. The polynomials $P(s), Q(s)$ must then have a special form (determined by the number k).

Theorem 3. Let two strictly increasing sequences of positive numbers $\{\lambda_n\}$, $\{\mu_n\}$, tending to infinity, be given, and suppose that

$$\underline{\lim} (\mu_{n+1} - \mu_n) \geq h > 0.$$

Denote

$$D^+ = \overline{\lim} \frac{n}{\lambda_n} \quad (D^+ < \infty).$$

Let ε be any positive number. Then the number of linearly independent solutions of equation (3) belonging to the label $\{\lambda_n, \mu_n\}$ does not exceed the number of λ_n 's lying in any interval (a, b) of the positive axis of length $b - a = D^+ + \varepsilon$.

Theorem 4. Let

$$\left\{ f(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}, \quad g(s) = \sum_{n=1}^{\infty} \frac{b_n}{\mu_n^s} \right\}$$

be a solution of equation (3). Suppose

$$h_\mu = \underline{\lim} (\mu_{n+1} - \mu_n) > 0$$

and let r, k be positive integers such that

$$\lambda_{r+k+1} - \lambda_r > 2h_\mu^{-1}.$$

Then $k \geq 2$, and each λ_n has the form

$$\lambda_n = m_1^{(n)} \lambda_{r+1} + m_2^{(n)} \lambda_{r+2} + \cdots + m_k^{(n)} \lambda_{r+k},$$

where $m_i^{(n)}$ are integers.

Further, if q is the least integer such that $\lambda_{q+1} > h_\mu^{-1}$, then each λ_n has the form $\lambda_n = p_1^{(n)} \lambda_1 + p_2^{(n)} \lambda_2 + \cdots + p_q^{(n)} \lambda_q$, where $p_i^{(n)}$ are integers.

From Theorem 4, with the aid of Theorems 1 and 2, one can obtain some corollaries. We give one of them.

Corollary. If the solution $\{f(s), g(s)\}$ of equation (3) is such that $\lim(\mu_{n+1} - \mu_n) = h_\mu > 0$ and $\lambda_1 < h_\mu^{-1} < \lambda_2$, then $f(s) = \alpha^s \zeta(s + 2k) [2^{1-(s+2k)} - 1]$, where k is an integer and α is a positive number.

§ 3. Theorems 1, 2, 3, 4 are derived from the following theorem:

Theorem A. If

$$\left\{ f(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}, \quad g(s) = \sum_{n=1}^{\infty} \frac{b_n}{\mu_n^s} \right\}$$

form a solution of equation (3), then the equality

$$\sum_{n=1}^{\infty} \sum_{k=0}^m \frac{\alpha_k a_n \lambda_n^{2k} t^l}{(t^2 + \lambda_n^2)^{k+1}} + K(t) = \sum_{j=0}^l \beta_j t^j \frac{d^j}{dt^j} F(t). \quad (\text{A})$$

holds.

Here $K(t)$ is a function defined and regular for all t for which $\log t$ is defined, and moreover $|K(t)| = O(|t|^{1-\eta})$ uniformly in the angle $|\arg t| \leq \theta_0$ for any θ_0 and some $\eta > 0$;

$$F(t) = \sum_{n=1}^{\infty} b_n e^{-2\pi\mu_n t} \quad (\text{for } \operatorname{Re} t > 0);$$

α_k are coefficients depending on $P(s)$; β_j are coefficients depending on $Q(s)$.

Remark. If the series in the left-hand side of (A) does not converge absolutely, then one must first make in (3) the substitution $s = s' + 2p$, where p is a sufficiently large integer.

For the proof of Theorems 1 and 2 it is necessary to take from both sides of (A) the finite difference of order l and then use the periodicity of the right-hand side of the equality obtained.

For the proof of Theorem 3 it is necessary to regard (A) as a differential equation with respect to $F(t)$ and to apply Mandelbrojt's theorem ⁽³⁾ on the singularities of the series $F(t)$ on the segment of the axis of convergence. A stronger form of the theorem is obtained if one applies another theorem of Mandelbrojt ⁽⁴⁾. Finally, for the proof of Theorem 4 a theorem of the type of Agmon's theorem ⁽⁵⁾, but somewhat more general, is needed.

Definition. We shall say that the function $F(s)$ has at the point s_0 a singularity of type k if the equality

$$F(s) = \frac{r_k}{(s - s_0)^k} + \frac{r_{k-1}}{(s - s_0)^{k-1}} + \dots + \frac{r_1}{s - s_0} + q(s) \ln(s - s_0) + r_0 + F_1(s),$$

holds, where $r_k \neq 0$ ($k \geq 0$); $q(s)$ is a polynomial; $F_1(s)$ is regular and vanishes at the point $s = s_0$.

Theorem B. Let $F(s)$ be a function representable for $\sigma > \sigma_c$ by the series

$$F(s) = \sum_{n=1}^{\infty} p_n(s) e^{-\mu_n s},$$

where $\lim(\mu_{n+1} - \mu_n) \geq h > 0$; $p_n(s) = a_n^{(l)} s^l + a_n^{(l-1)} s^{l-1} + \dots + a_n^{(0)}$ is a polynomial of degree not exceeding l ; $a_n^{(j)} = o(a_n^{(m)})$ for $j \neq m$, $n \rightarrow \infty$. Suppose that on some segment L of the axis of convergence of length greater than $4\pi h^{-1}$

there are only isolated singularities of $F(s)$ —the points $\sigma_c + \alpha_1 i$, $\sigma_c + \alpha_2 i$, ..., $\sigma_c + \alpha_q i$, and some segment L_1 of the axis of convergence of length greater than $2\pi h^{-1}$, not containing the point $s = \sigma_c$, contains only isolated singularities of $F(s)$ of type k or of lower type.

Then, for any singular point of type k on the axis of convergence $s = \sigma_c + \alpha i$, the equality

$$\alpha = m_1^{(\alpha)} \alpha_1 + m_2^{(\alpha)} \alpha_2 + \dots + m_q^{(\alpha)} \alpha_q$$

holds, where $m_j^{(\alpha)}$ are integers. Moreover, if $\sigma_c + \alpha_n i$ is a singularity on L of type lower than k , then one may assume that $m_n^{(\alpha)} = 0$.

I take this opportunity to express my deep gratitude to Prof. A. O. Gelfond for a number of valuable pieces of advice and to Prof. I. K. Andronov for his attention to my work.

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Received
11 IX 1959

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Note: Figure translations are in progress. See original paper for figures.

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