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Abstract

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ON THE REPRESENTATION OF INFINITE-DIMENSIONAL COMPACTA AS AN INVERSE LIMIT OF POLYHEDRA

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Freudenthal proved ([1]) that every compactum X can be represented as such an inverse limit of a sequence of polyhedra $\{P_i, f_i^j\}$, in which the mappings f_i^j are piecewise-affine irreducible* mappings "onto." Moreover, in order that the compactum be n-dimensional, it is necessary and sufficient that it be representable as an inverse limit of a sequence of n-dimensional polyhedra in which the mappings f_i^j satisfy the conditions stated above.

The aim of the present note is to give an analogous characterization for one class of infinite-dimensional compacta, namely compacta that are the sum of a countable number of their closed finite-dimensional subsets. For brevity we shall call such compacta weakly countable-dimensional.**

Theorem 1. *In order that a compactum X be weakly countable-dimensional, it is necessary and sufficient that it can be represented as an inverse limit of a sequence of polyhedra $\{P_i, f_i^j\}$ such that the mappings f_i^j satisfy Freudenthal's conditions and, for every thread $\xi = \{\xi_i\}$, $\xi_i \in P_i$, the dimensions of the carriers $T(\xi_i)$ are bounded in the aggregate.*

In proving this theorem we shall, in the main outlines, follow Freudenthal.

The proof of necessity is based on the following proposition: the space X is weakly countable-dimensional if and only if there exists a sequence of starwise inscribed in one another, shrinking***, finite open coverings such that at each point $x \in X$ the orders of all the coverings are bounded in the aggregate ([3,4]). Let $\{\alpha_n\}$ be a system of such coverings of the compactum X. We shall construct a sequence of polyhedra P_i and mappings $g_i : X \to P_i$, and also

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* A mapping is called irreducible if, after an admissible deformation, it remains a mapping "onto." Here a deformation of a mapping is called admissible if, in the process of deformation, the image of each point does not leave its carrier. Finally, the carrier of a point ξ in a given complex is the closed simplex of least dimension containing this point; we shall denote it by $T(\xi)$. Irreducibility of a

mapping is equivalent to the fact that, for every closed simplex, its full preimage is mapped onto it essentially.

** Countable-dimensional spaces are those that are countable sums of their zero-dimensional (not necessarily closed) subsets. An example of a countable-dimensional but not weakly countable-dimensional compactum was constructed by Yu. M. Smirnov ([2]).

*** A covering β is starwise inscribed in a covering α if the star of any element of β with respect to β is contained in some element of α. The star of a set with respect to a covering is the sum of the elements of the covering that intersect the set. A sequence of coverings is called shrinking if, for every point and every neighborhood of it, there is a covering in the sequence such that the star of the point with respect to it is contained in the given neighborhood.

maps $f_i^j : P_j \to P_i$ such that the following conditions are satisfied:
1) $\dim T(g_i x) \leq k(x)$ for every point $x \in X$, where $k(x)$ is the upper bound of the multiplicities of the coverings α_n at the point x; 2) the mappings g_i are irreducible mappings "onto" ; 3) the mappings f_i^j are piecewise affine; 4) $T(f_i^j, g_j x) \subseteq T(g_i x)$ for each point $x \in X$; 5) for every simplex $T \in P_j$ the diameter of its image $f_i^j T$ does not exceed $1/2^{j-1}$.

For the construction of the polyhedron P_1 and the mapping g_1 we proceed as follows: first we construct the barycentric mapping of the compactum X into the nerve of the covering α_1, and then, applying the sweeping-out operation ([5]), we obtain an irreducible mapping g_1 onto some subcomplex P_1 of the nerve. If the polyhedra $P_1, \dots, P_r$ and the mappings g_i, f_i^j, $i \leq j \leq r$, satisfying the indicated conditions have already been constructed, then the construction of the polyhedron P_{r+1} and the mapping g_{r+1} is carried out as follows: we choose a sufficiently fine subdivision of the polyhedron P_r so that the diameters of the images of the simplexes of this subdivision under the mappings f_i^r do not exceed $1/2^r$. By compactness of the space X, in the sequence of coverings $\{\alpha_n\}$ there is a covering, say α_{r+1}, inscribed in the covering composed of the inverse images of the principal stars of the subdivision of the polyhedron P_r under the mapping g_r. Next, the polyhedron P_{r+1} and the mapping g_{r+1} are constructed from the covering α_{r+1} in exactly the same way as the polyhedron P_1 and the mapping g_1 were constructed from the covering α_1; in this process, as the mapping f_r^{r+1} we take the simplicial mapping of the polyhedron P_{r+1} into the subdivision of the polyhedron P_r, and, finally, put $f_i^{r+1} = f_i^r f_r^{r+1}$. It is easy to verify that the sequence obtained in this way satisfies all the conditions formulated above.

We shall show that the inverse-limit compactum P of the sequence $\{P_i, f_i^j\}$ is homeomorphic to X. To this end we construct mappings $f_i : X \to P_i$ such that $f_i = f_i^j f_j$. For each mapping f_i we take the limit of the sequence of mappings $f_i^j g_j$ (i fixed), which converges uniformly, since

$$\rho(f_i^{j+1} g_{j+1} x, f_i^j g_{jx}) = \rho(f_i^j f_j^{j+1} g_{j+1} x, f_i^j g_{jx}) \leq 1/2^{j-1},$$

which in turn follows from the fact that g_{jx} and $f_j^{j+1}g_{j+1}x$ belong to one and the same closed simplex. Since the mapping f_i can be obtained from g_i by means of an admissible deformation, f_i is an irreducible mapping onto the polyhedron P_i. Assigning to each point $x \in X$ the thread $\{f_i x\}$, we obtain the desired homeomorphism of the compactum X onto P.

It remains to verify that for each thread $\xi = \{\xi_i\}$, $\xi_i \in P_i$, the dimensions of the carriers $T(\xi_i)$ are bounded in the aggregate. But this follows from the fact that every thread ξ has the form $\{f_i x\}$, $x \in X$, while $T(f_i x) \leq T(g_i x)$ and $\dim T(g_i x) \leq k(x)$.

The proof of sufficiency can also be obtained from the theorem on coverings formulated above: in the inverse-limit compactum, as the sequence of coverings one may take the coverings composed of the inverse images of the principal stars of the complexes P_i. However, it is easy to give a direct proof as well. For any natural number n, consider the following sequence of closed subsets $X_{i,n}$ of the polyhedra P_i: let $X_{1,n}$ be the n-dimensional skeleton of the polyhedron P_1, and suppose the sets $X_{i,n}$ for $i \leq r$ have already been constructed; put

$$X_{r+1,n} = (f_r^{r+1})^{-1} X_{r,n} \cap P_{r+1}^n,$$

where P_{r+1}^n denotes the n-dimensional skeleton of the polyhedron P_{r+1}. The inverse limit X_n of the sequence $\{X_{i,n}, f_i^j\}$ with the same mappings f_i^j (considered on $X_{i,n}$) is naturally embedded in the compactum X. From the condition imposed on the threads it follows directly that

$$X = \bigcup_n X_n.$$

The sets X_n are compact and $\dim X_n \leq n$. Thus the theorem is completely proved.

Theorem 2. If a weakly countable-dimensional compactum X is represented as the inverse limit of a sequence of polyhedra $\{Q_k, h_k^l\}$ satisfying Freudenthal' s conditions, then by passing to a subsequence, by an admissible deformation of the maps h_k^l, and by subdividing the complexes Q_k, one can pass from the original sequence to a sequence whose limit is still equal to X, and whose nerves satisfy the condition of Theorem 1.

The proof of this theorem follows directly from the following proposition of Freudenthal: if a compactum X is represented as the inverse limit of two sequences of polyhedra $\{P_i, f_i^j\}$ and $\{Q_k, h_k^l\}$, then in them one can choose subsequences P_{i_n} and Q_{k_n} and construct maps

$$h_{2n-1} : Q_{k_n} \to P_{i_n} \quad \text{and} \quad h_{2n} : P_{i_{n+1}} \to Q_{k_n}$$

in such a way that these maps will be simplicial with respect to certain subdivisions of the complexes P_{i_n} and Q_{k_n}, the compositions $h_{2n}h_{2n+1}$ and $h_{2n-1}h_{2n}$

are obtained by an admissible deformation from the maps $h_{k_n}^{k_{n+1}}$ and $f_{i_n}^{i_{n+1}}$, and the limit of the alternating sequence of polyhedra

$$P_{i_n}, Q_{k_n}, \qquad n = 1, 2, \dots,$$

is equal to X.

As $\{P_i, f_i^j\}$ we take the sequence constructed in the preceding theorem, and apply Freudenthal' s proposition to the sequences $\{P_i, f_i^j\}$ and $\{Q_k, h_k^l\}$. From the resulting alternating sequence we choose the subsequence consisting of the polyhedra Q_{k_n}, and take in them those same subdivisions with respect to which the maps h_{2n} are simplicial. This sequence will be the desired one.

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Note: Figure translations are in progress. See original paper for figures.