ASYMPTOTICS OF THE GREEN FUNCTION FOR PETROVSKII-WELL-POSED EQUATIONS WITH MANY VARIABLES

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Abstract

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M. V. FEDORYUK

ASYMPTOTICS OF THE GREEN FUNCTION FOR PETROVSKII-WELL-POSED EQUATIONS WITH MANY VARIABLES

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In the present note we find the asymptotics of the fundamental solution of the Cauchy problem as $t \to +0$, $|x| \to \infty$ for a broad class of Petrovskii-well-posed equations with constant coefficients and many spatial variables. For the case of one spatial variable the asymptotics was found in ([1]). The asymptotics obtained is also valid as $t \to +0$, x fixed, and as $|x| \to \infty$, t fixed.

1. Let us denote:

$$x = (x_1, \dots, x_k) = \rho(\alpha_1, \dots, \alpha_k), \qquad \rho > 0, \qquad \sum_{j=1}^{k} \alpha_j^2 = 1,$$

$\alpha = (\alpha_1, \dots, \alpha_k)$, $k \geq 2$; α_j, x_j are henceforth real throughout. Consider the Cauchy problem

$$\frac{\partial u}{\partial t} = P\left(i\frac{\partial}{\partial x}\right)u, \tag{1}$$

$$u\Big|_{t=0} = \delta(x), \tag{2}$$

where $P\left(i\frac{\partial}{\partial x}\right)$ is a differential operator with constant coefficients. The solution of the Cauchy problem (1), (2) will henceforth be denoted everywhere by G.

Equation (1) is called **Petrovskii-well-posed** ([2]) if $\operatorname{Re} P(s) < C$ for real s. Decompose $P(s)$ into a sum of homogeneous polynomials:

$$P(s) = P_n(s) + P_{n-1}(s) + \dots + P_0,$$

where the degree of $P_j(s)$ is j.

We introduce the following classification of Petrovskii-well-posed equations:

1°. Equation (1) is **parabolic in the sense of Petrovskii** if

$$\operatorname{Re} P_n(s) < 0 \quad \text{for} \quad \sum_{j=1}^{k} \sigma_j^2 = 1, \qquad s_j = \sigma_j + i\tau_j.$$

2°. Equation (1) is **properly parabolic in the sense of Shilov** if

$$\operatorname{Re} P(\sigma) < C_1 - C_2|\sigma|^h, \qquad h < n, \qquad C_2 > 0.$$

3°. Equation (1) is **properly Petrovskii-well-posed** in all remaining cases.

We shall restrict ourselves to the study of the following classes of equations:

1′. Parabolic in the sense of Petrovskii.

2′. Properly parabolic in the sense of Shilov, for which $P_n(s)$, $P_{n-1}(s), \dots, P_{p+1}(s)$ have purely imaginary coefficients; $P_n(s)$ is nondegenerate; $\operatorname{Re} P_p(s) < 0$ for

$$\sum_{j=1}^{k} \sigma_j^2 = 1, \qquad p > 0.$$

3′. Properly Petrovsky well-posed equations, for which $P_n(s), P_{n-1}(s), \dots, P_1(s)$ have purely imaginary coefficients; $P_n(s)$ is nonzero. Suppose, moreover, that system (8) has no multiple roots.

Theorem 1. For a Petrovsky parabolic equation, as $\rho \to +\infty,\ t \to +0$,

$$G(x,t) \sim \sum_{j=1}^{m} \exp\left[\frac{\rho^{\frac{n}{n-1}}}{t^{\frac{1}{n-1}}} \sum_{l=0}^{\infty} c_{lj}(\alpha)\left(\frac{t}{\rho}\right)^{\frac{l}{n-1}}\right] A_j(\alpha,\rho,t), \tag{3}$$

where

$$A_j(\alpha,\rho,t) = \rho^{-\frac{k(n-2)}{2(n-1)}} t^{-\frac{k}{2(n-1)}} B_j(\alpha), \tag{4}$$

$$\operatorname{Re} c_{0j}(\alpha) < a < 0, \qquad m < (n-1)^k. \tag{5}$$

Theorem 2. For equations of types 2′ and 3′, where $i \cdot P_n(s)$ is definite, the asymptotics of $G(x,t)$ as $\rho \to +\infty,\ t \to +0$ are determined by formulas (3), (4), where in case 3′

$$c_{lj}(\alpha) \text{ are purely imaginary}, \qquad j = 1, \dots, m, \qquad l = 0, 1, \dots; \tag{6}$$

in case 2′

$$\operatorname{Re} c_{0j}(\alpha) = \operatorname{Re} c_{1j}(\alpha) = \dots = \operatorname{Re} c_{p+1,j}(\alpha) = 0, \qquad \operatorname{Re} c_{pj}(\alpha) < 0, \tag{7}$$

$$j = 1, \dots, m.$$

Theorem 3. Suppose that $P_n(s) \cdot i$ is indefinite and equation (1) has type $2'$. Then the real sphere $\Omega: \sum_{j=1}^{k} \alpha_j^2 = 1$ consists of two parts Ω_{I} and Ω_{II}; Ω_{II} is the set of points $\alpha \in \Omega$ for which the system

$$\frac{\partial P_n}{\partial s_j} = i\alpha_j, \qquad j = 1, \dots, k, \tag{8}$$

has real solutions. As $\rho \to +\infty,\ t \to +0$, the asymptotics of $G(x,t)$ are determined by formulas (3), (4), (5), if $\alpha \in \Omega_{\text{I}}$, and by formulas (3), (4), (6), if $\alpha \in \Omega_{\text{II}}$.

Theorem 4. Suppose that $P_n(s) \cdot i$ is indefinite and equation (1) has type $3'$. Then

$$Q_q\left(\frac{\partial}{\partial x}\right) G(x,t) = (\Delta - 1)^{\left[\frac{q+k}{2}\right]+2} G_0(x,t), \tag{9}$$

$$R_r\left(\frac{\partial}{\partial t}\right) G(x,t) = (\Delta - 1)^{\left[\frac{rn+k}{2}\right]+2} G_0'(x,t), \tag{10}$$

where $\Delta = \partial^2/\partial x_1^2 + \dots + \partial^2/\partial x_k^2$, and $Q_q(\partial/\partial x)$ and $R_r(\partial/\partial t)$ are differential operators with constant coefficients of orders q and r, respectively; G_0 and G_0' are continuous functions.

The asymptotics of G_0 and G_0' as $\rho \to +\infty,\ t \to +0$ are determined by formulas (3), (6), if $\alpha \in \Omega_{\text{II}}$, where

$$A_j^q(\alpha,\rho,t) = \rho^{-\frac{k(n-2)}{2(n-1)} - \frac{2\left[\frac{k+q}{2}\right]+4-q}{n-1}} t^{-\frac{k}{2(n-1)} + \frac{2\left[\frac{k+q}{2}\right]+4-q}{n-1}} B_j^{(q)}(\alpha), \tag{11}$$

$$A_j^{\prime(r)}(\alpha,\rho,t) = \rho^{-\frac{k(n-2)}{2(n-1)} - \frac{2\left[\frac{k+nr}{2}\right]+4-nr}{n-1}} t^{-\frac{k}{2(n-1)} + \frac{2\left[\frac{k+nr}{2}\right]+4-nr}{n-1}} B_j^{\prime(r)}(\alpha). \tag{12}$$

If $\alpha \in \Omega_{\text{II}}$, then for G_0 and G_0' the estimate

$$\begin{aligned} |G_0| &< C(\alpha) \exp\left[-C_1(\alpha)\rho^{\frac{n}{n-1}}/t^{\frac{1}{n-1}}\right], \\ |G_0'| &< C'(\alpha) \exp\left[-C_1'(\alpha)\rho^{\frac{n}{n-1}}/t^{\frac{1}{n-1}}\right]. \end{aligned} \tag{13}$$

is valid.

Let us give one more, more precise theorem for the case of indefinite $iP_n(s)$.

Theorem 5. Suppose the conditions of Theorem 4 are satisfied and, moreover, on every ray $s_j = \sigma_j^0 \tau,\ 0 \leqslant \tau < \infty$, where the σ_j^0 are real, for $|s| > a(\sigma)$

$$|P_n(s)| > C(\sigma)\tau^2. \qquad (14)$$

Then $G(x,t)$, for $t > 0$, is an entire function of x and an infinitely differentiable function of t. The asymptotics of $G(x,t)$ as $\rho \to +\infty,\ t \to +0$ is determined by formulas (3), (4), (5), if $\alpha \in \Omega_{\mathrm{I}}$, and by formulas (3), (4), (6), if $\alpha \in \Omega_{\mathrm{II}}$.

Corollary. From the asymptotics obtained for $G(x,t)$, classes of correctness for the solution of the Cauchy problem are easily obtained, and in a number of cases broader ones than those obtained in (3). For example, under the conditions of Theorem 3, if Ω_{I} is nonempty, the class of correctness is the class of functions

$$|u_0(x_1,\dots,x_k)| < Ce^{C_1\rho^{\frac{n}{n-1}-\varepsilon}}, \qquad -\alpha \in \Omega_{\mathrm{I}},\ C_1 > 0;$$

$$|u_0(x_1,\dots,x_k)| < C'e^{C_1'\rho^{\frac{p}{n-1}-\varepsilon}}, \qquad -\alpha \in \Omega_{\mathrm{II}},\ C_1' > 0.$$

Remark 1. In Theorems 1–3, $G(x,t)$ is, for $t > 0$, an entire function of x and an infinitely differentiable function of t. Under the conditions of Theorem 4 this is, generally speaking, not so, as is shown by the example of S. L. Sobolev[4].

Remark 2. The asymptotics of the derivatives of $G(x;t)$ with respect to x and t is obtained by simple differentiation of formula (3) in Theorems 1, 2, 3, and 5.

Remark 3. Analogous results can be obtained for any equations correct in the sense of Petrovskii, and also for equations with coefficients depending on t.

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Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.