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Abstract

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ON SOME TRANSFORMATIONS OF MARKOV PROCESSES

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In the present paper one general class of transformations of Markov processes is introduced and studied; it contains, as special cases, a number of special transformations studied earlier (the formation of a subprocess [1], the transformation of a Wiener process leading to the appearance of drift (see, for example, [2], and others)). In constructing the aforementioned general class of transformations, the notion of a multiplicative functional of a process plays the basic role.*

We use the terminology and notation of the monograph [1].

1. Let $X = (x_t, \zeta, \mathcal{M}_t^s, \mathbf{P}_{s,x})$ be a Markov process in a measurable space $(E, \mathfrak{B})$ and on the time interval $[0, T)$. For an arbitrary finite measure μ on the σ-algebra $\mathfrak{B}$ we put

$$\mathbf{P}_{s,\mu}(\mathfrak{B}) = \int_E \mathbf{P}_{s,x}(B)\mu(dx) \qquad (B \in \mathcal{N}^s)$$

($\mathcal{N}^s$ is the σ-algebra in the space of elementary events Ω, generated by the sets $\{\omega : x_t(\omega) \in \Gamma\}$ $(t \geq 0)$, $\Gamma \in \mathfrak{B}$). Further, we put $A \in \overline{\mathcal{N}}^s$, if for every finite measure μ one can construct A_1, A_2 from $\mathcal{N}^s$ such that $A_1 \subseteq A \subseteq A_2$ and $\mathbf{P}_{s,\mu}(A_1) = \mathbf{P}_{s,\mu}(A_2)$.

A nonnegative function $\alpha_t^s(\omega)$ $(0 \leq s \leq t < \zeta(\omega))$ is called a **multiplicative functional** of the Markov process X, if the following conditions are fulfilled:

1 A. α_t^s is $\mathcal{M}_t^s \cap \overline{\mathcal{N}}^s$-measurable.

1 B. $\alpha_t^s(\omega)\alpha_u^t(\omega) = \alpha_u^s(\omega)$ $(0 \leq s \leq t \leq u < \zeta(\omega))$.

We give some examples of multiplicative functionals.

a) If $g(u, x)$ is any $\mathfrak{B}_T^0 \times \mathfrak{B}$-measurable function ($\mathfrak{B}_T^0$ denotes the σ-algebra of all Borel subsets of the interval $[0, T]$), then $\alpha_t^s = g(t, x_t)/g(s, x_s)$ is a multiplicative functional.

b) Let the function $\tau_s(\omega)$ $(0 \leq s < \zeta(\omega))$ be subject to the conditions:

$$s \le \tau_s(\omega) \le \zeta(\omega); \qquad \{\tau_s > t\} \in \mathcal{M}_t^s \cap \overline{\mathcal{N}}^s; \qquad \{\tau_s > t\} \subseteq \{\tau_s = \tau_t\} \quad (0 \le s \le t < T).$$

Then $\alpha_t^s = \chi_{\tau_s > t}$ is a multiplicative functional (the symbol χ_A denotes the characteristic function of the set A).

c) Let $V(u,x)$ be a $\mathfrak{B}_T^0 \times \mathfrak{B}$-measurable function, μ a measure on $\mathfrak{B}_T^0$, and let the integral

$$\int_s^\zeta V(u, x_u)\mu(du)$$

converge or diverge to $+\infty$ for all

* A brief visual description of this class of transformations (for homogeneous Markov processes) is contained in the survey article [3]. Proofs of the results formulated in the present note will be published in the *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability.*

$0 \le s < \zeta(\omega)$. Then

$$\alpha_t^s = \exp\left[-\int_{(s,t]} V(u,x_u)\mu(du)\right] \qquad (0 \le s \le t < \zeta(\omega))$$

is a multiplicative functional.

d) Let $X = (x_t, T, \mathcal{M}_t^s, \mathbf{P}_{s,x})$ be an n-dimensional Wiener process given on the time interval $[0,T]$. Let $f(u,x)$ $(u \in [0,T], x \in E)$ be a function with values in the n-dimensional space E, satisfying the conditions: for every $\Gamma \in \mathcal{B}$, $\{(u,x) : f(u,x) \in \Gamma\} \in \mathcal{B}_T^0 \times \mathcal{B}$; for any $t \in [0,T)$,

$$\sup_{0 \le u \le t,\ x \in E} f^2(u,x) < \infty^*.$$

It is proved in ([4]) that one can choose the value of the stochastic integral so that the formula

$$\alpha_t^s = \exp\left[-\int_s^t f(u,x_u)\,dx_u\right] \qquad (0 \le s \le t < \zeta(\omega))$$

defines a continuous** multiplicative functional.

2. **Theorem.** Let $X = (x_t, \zeta, \mathcal{M}_t^s, \mathbf{P}_{s,x})$ be a normal*** Markov process, given on the time interval $[0,T)$ and having elementary-event space Ω. Let α_t^s be a multiplicative functional of the process X. If $\mathbf{M}_{s,x}\alpha_t^s = 1$ for all $0 \le s \le t < T,\ x \in E$, then the formula

$$\widetilde{P}(s,x;t,\Gamma) = \mathbf{M}_{s,x}[\chi_\Gamma(x_t)\alpha_t^s] \tag{1}$$

defines a certain transition function.

In order that there exist a Markov process with transition $\widetilde{X}$-function (1), for which each trajectory is obtained by terminating some trajectory of the process X, it is sufficient that some nonnegative function $\xi_t(\omega)$ $(0 \le t < \zeta(\omega))$ satisfy the conditions:

2A. $\alpha_t^s \xi_t \le \xi_s$ $(0 \le s \le t < \zeta(\omega))$.

2B. $\lim_{t\downarrow s} \alpha_t^s \xi_t = \alpha_s^s \xi_s$.

2C. ξ_t is $\mathcal{N}_t$-measurable.

2D. $\mathbf{M}_{s,x}\xi_s = 1$ for arbitrary $0 \le s < T$ and $x \in E$.

The process $\widetilde{X}$ can be constructed in the following way. Put

$$\widetilde{\Omega} = \Omega \times [0,T], \qquad \widetilde{\mathcal{M}}^s = \mathcal{M}^s \times \mathcal{B}_T^0,$$

$$\tilde{\zeta}(\omega,u) = \min[\zeta(\omega),u], \qquad \tilde{x}_t(\omega,u) = x_t(\omega) \quad \text{for } 0 \le t < \tilde{\zeta}(\omega,u),$$

and denote by $\widetilde{\mathcal{M}}_t^s$ the totality of all subsets of the space $\widetilde{\Omega}$ having the form $A \times (t,T]$, where $A \in \mathcal{M}_t^s$. We have defined all elements of the process $\widetilde{X} = (\tilde{x}_t, \tilde{\zeta}, \widetilde{\mathcal{M}}_t^s, \widetilde{\mathbf{P}}_{s,x})$, with the exception of the measures $\widetilde{\mathbf{P}}_{s,x}$. The latter are defined as follows. Put $\psi_t^s = \alpha_t^s \xi_t$ $(0 \le s \le t < \zeta(\omega))$. Let $\omega \in Q_s = \{\omega : \alpha_s^s(\omega) = 1\}$. By virtue of conditions 2A and 2B, ψ_t^s is a nonincreasing right-continuous function of t for $t \in [s, \zeta(\omega))$. Therefore there exists, and moreover a unique, measure ψ^s on the σ-algebra $\mathcal{B}_T^0$, concentrated on the interval $(s,\zeta(\omega)]$, and such that $\psi^s(t,\zeta] = \psi_t^s$. For $\omega \in \bar{Q}_s$ denote by ψ^s the unit measure concentrated at the point s.

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* By f^2 one should understand the scalar square of the vector f.

** The functional α_t^s is called continuous if, for any $0 \le s < \zeta(\omega)$, $\alpha_t^s(\omega)$ is a continuous function of t on $[s, \zeta(\omega))$.

*** A Markov process is called normal if $\mathbf{P}_{s,x}\{\zeta > s\} = 1$ for all $s \in [0,T),\ x \in E$.

The measure $\widetilde{\mathbf{P}}_{s,x}$ is defined by the formula

$$\widetilde{\mathbf{P}}_{s,x}(C) = \mathbf{M}_{s,x}\psi(C_\omega), \tag{2}$$

where C_ω denotes the ω-section of the set C, i.e., the collection of numbers u such that $(\omega, u) \in C$.

The Markov process constructed by us,

$$\widetilde{X} = (\tilde{x}_t, \xi, \widetilde{\mathcal{M}}_t, \widetilde{\mathbf{P}}_{s,x}),$$

we propose to call the (α_t^s, ξ_t)-subprocess of the process X.

Let us emphasize once more that all elements of $\widetilde{X}$, except $\widetilde{\mathbf{P}}_{s,x}$, do not depend on α_t^s and ξ, while the transition function $\widetilde{P}(s, x; t, \Gamma)$ of the process $\widetilde{X}$ does not depend on ξ_t.

3. Let us consider the most important special classes of (α_t^s, ξ_t)-subprocesses. We call a (α_t^s, ξ_t)-subprocess an α_t^s-subprocess if $\xi_t(\omega) = 1$ for all $0 \leq t < \zeta(\omega)$. In order for it to be possible to form an α_t^s-subprocess, it is necessary and sufficient that the multiplicative functional α_t^s satisfy the conditions

$$\alpha_t^s(\omega) \leq 1 \quad (0 \leq s \leq t < \zeta(\omega)), \qquad \lim_{t \downarrow s} \alpha_t^s(\omega) = \alpha_s^s(\omega).$$

(These conditions are satisfied, for example, for the functional 1) and for the functional 1) when $V \geq 0$.)

If one identifies the subsets $A \times [0, T]$ of the space $\widetilde{\Omega}$ with the subsets A of the space Ω, then in the case of an α_t^s-subprocess it turns out that the measure $\widetilde{\mathbf{P}}_{s,x}$ is an extension of the measure $\mathbf{P}_{s,x}$. Hence it easily follows that

$$\alpha_t^s = \widetilde{\mathbf{P}}_{s,x}\{\tilde{\zeta} > t \mid \mathcal{M}^s\} = \widetilde{\mathbf{P}}_{s,x}\{\tilde{\zeta} > t \mid \widetilde{\mathcal{M}}_t^s\} \quad \text{almost surely } (\Omega_t, \widetilde{\mathbf{P}}_{s,x}). *$$

Therefore the intuitive picture of the formation of an α_t^s-subprocess ** may be described as follows: the trajectories of the original process are cut off with a certain probability distribution, and $\alpha_t^s(\omega)$ denotes the conditional probability that the trajectory $x_u(\omega)$ will not be cut off during the time interval $[s, t]$ (provided all the phenomena connected with the process X during the time $[s, t]$ or during the time $[s, T]$ are known).

4. Let the multiplicative functional α_t^s and the function ξ_t^s satisfy conditions 2 –2 , as well as the conditions:

4 . $\alpha_t^{s\xi_t} = \alpha_s^{s\xi_s}$ $(0 \leq s \leq t < \zeta(\omega))$; $\mathbf{P}_{s,x}\{\chi_s^s = 1\} = 1$ for all $0 \leq s < T,\ x \in E$.

Then, as is easy to see, conditions 2 –2 are fulfilled, and one may form the (α_t^s, ξ_t)-subprocess of the process X. For this subprocess

$$\widetilde{\mathbf{P}}_{s,x}\{\tilde{\zeta} = \zeta\} = 1$$

for all $s \in [0, T),\ x \in E$.

According to Theorem 2.5 ([1]), the mapping $\gamma : \Omega \to \widetilde{\Omega}$, defined by the formula $\gamma(\omega) = (\omega, \zeta(\omega))$, specifies a transformation of the elementary-event space of the process Ω. After this transformation we obtain the process

$$X' = (x_t, \zeta, \mathcal{M}_t, \mathbf{P}'_{s,x}),$$

all elements of which, with the exception of $\mathbf{P}'_{s,x}$, are the same as for the process X, while the measures $\mathbf{P}'_{s,x}$ can be obtained from the measures $\mathbf{P}_{s,x}$ by the formula

$$\mathbf{P}'_{s,x}(C) = \int_C \xi_s(\omega) \mathbf{P}_{s,x}(d\omega). \tag{3}$$

Suppose that the multiplicative functional α_t^s is subject to the condition

$$\mathbf{P}_{s,x}\{\alpha_s^s = 1\} = 1 \quad (s \in [0, T),\ x \in E),$$

and that for each $\omega \in \Omega_s$ there exist–

* We put $\Omega_t = \{\omega : \zeta(\omega) > t\}$.

** α_t^s-subprocesses of Markov processes are studied in detail in ([1]), Ch. 3.

there exists the limit $\alpha_{\zeta-0}^s = \lim_{t \uparrow \zeta} \alpha_t^s$ and $\mathbf{M}_{s,x} \alpha_{\zeta-0}^s = 1$ for all $s \in [0, T)$, $x \in E$. It is easy to see that in this case the functions $\xi_t = \alpha_{\zeta-0}^t$ satisfy conditions 4A, 2B, 2Г. Therefore one can transform the measures of the process by formula (3), putting $\xi_s = \alpha_{\zeta-0}^s$. What has been said applies, in particular, to the functional

$$\alpha_t^s = \exp\left[-\frac{1}{2}\int_s^t f^2(u, x_u)\, du - \int_s^t f(u, x_u)\, du\right],$$

where $f(u, x)$ is the function described in 1), for which

$$\sup_{0 \le u < T,\ x \in E} f^2(u, x) < \infty$$

(see in more detail ([4])).

5. The third important class of (α_t^s, ξ_t)-subprocesses is connected with functions $\eta_t(\omega)$ $(0 \le t < \zeta(\omega))$ satisfying the conditions:

5A. $\eta_t(\omega)$ is $\mathcal{N}^t$-measurable.

5B. $0 \leq \eta_t(\omega) \leq \eta_s(\omega)$ for $0 \leq s \leq t < \zeta(\omega)$.

5 . $\eta_{t+0}(\omega) = \eta_t(\omega)$ $(0 \leq t < \zeta(\omega))$.

Put $f(t,x) = \mathbf{M}_{t,x}\eta_t$. Suppose that $0 < f(t,x) < \infty$ for all $t \in [0,T)$, $x \in E$. It is easy to see that the pair $\alpha_t^s = f(t,x_t)/f(s,x_s)$, $\xi_t = \eta_t/f(t,x_t)$ satisfies requirements 2A–2 . Therefore this pair corresponds to a certain (α_t^s, ξ_t)-subprocess. Its transition function is given by the formula

$$\widetilde{P}(s,x;t,\Gamma) = \frac{1}{f(t,x)} \int_\Gamma P(s,x;t,dy)\, \tilde{f}(t,y).$$

6. Suppose now that the process X is homogeneous. A functional α_t^s of X is called homogeneous if for any $h \geq 0$, $0 \leq s \leq t < T$ and $x \in E$,

$$\theta_h \varphi_t^s = \varphi_{t+h}^{s+h}$$

(almost surely Ω_{t+h}, $\mathbf{P}_{s+h,x}$). (The functionals 1a), 1b), 1) are homogeneous if the corresponding functions $g, V, \tilde{f}$ do not depend on u.)

It is proved that if α_t^s is a homogeneous multiplicative functional of a homogeneous Markov process X, then every (α_t^s, ξ_t)-subprocess is a homogeneous Markov process.

7. Let X be a homogeneous Markov process. A function η is called an excessive random variable for the process X if η is $\mathcal{N}^t$-measurable, $\theta_t\eta \leq \eta$ for all $t \geq 0$ and $\lim_{t\downarrow 0} \theta_t \eta = \eta$. If η is an excessive random variable, then $\eta_t = \theta_t\eta$ satisfies conditions 5A–5 , and we can construct the (α_t^s, ξ_t)-subprocess described in § 5. By virtue of § 6 this subprocess will be a homogeneous Markov process. Its transition function is given by the formula

$$\widetilde{P}(t,x,\Gamma) = \frac{1}{f(x)} \int_\Gamma P(t,x,dy) f(y), \tag{4}$$

where $f(x) = \mathbf{M}_x\eta$. The function $f(x)$ is excessive in the sense of Hunt ([5]). Formula (4) makes it possible, from each homogeneous transition function $P(t,x,\Gamma)$ and each function f excessive with respect to it, to construct a new homogeneous transition function $\widetilde{P}(t,x,\Gamma)$. However, in the general case it is impossible to construct a process $\widetilde{X}$ with transition function $\widetilde{P}(t,x,\Gamma)$ for which each trajectory would be the beginning of some trajectory of the process X.

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Note: Figure translations are in progress. See original paper for figures.