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Abstract

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ON THE QUESTION OF THE SPATIAL MOTION OF AN AXISYMMETRIC RIGID BODY ABOUT A FIXED POINT UNDER THE ACTION OF MOMENTS SLOWLY VARYING IN TIME

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Consider an axisymmetric rigid body with a fixed point O, situated on its axis of symmetry. Let us denote (see Fig. 1) by Ox_1 a certain direction fixed in space, by Ox the axis of symmetry of the body, by Oy an axis lying in the plane Oxx_1 and directed perpendicular to the axis Ox, and by Oz an axis perpendicular to the plane Oxx_1. We shall consider the motion of the body under the action of a restoring moment $M_z(\tau, \theta)$, depending on the "slow" time $\tau = \varepsilon t$ (ε is a small parameter, t is time) and on the nutation angle θ, and directed along the axis Oz, and of small damping moments $\varepsilon M^{\omega x}(\tau)\omega_x$, $\varepsilon M^{\omega}(\tau)\omega_y$ and $\varepsilon M^{\omega}(\tau)\omega_z$, directed, respectively, along the axes Ox, Oy, and Oz. Here ω_x, ω_y, and ω_z denote the projections of the angular velocity of the body on the axes of the coordinate system $Oxyz$. The angular velocity of the body is the resultant of three rotations (see Fig. 1): rotation about the axis Oz with velocity $d\theta/dt$, rotation about the axis Ox_1 (precession) with velocity λ, and rotation about the axis Ox with velocity μ (proper rotation). Below we restrict ourselves to the solution of the most complicated part of the problem—the determination of the nutation angle θ and the precession velocity λ.

Fig. 1

The system of equations for determining these quantities (1) can be written in the form

$$\frac{d^2\theta}{dt^2} - \lambda^2 \sin\theta\cos\theta + r(\tau)\lambda\sin\theta + F(\tau,\theta) + \varepsilon f(\tau)\frac{d\theta}{dt} = 0, \qquad (1)$$

$$\frac{d}{dt}\left[\lambda\sin^2\theta + r(\tau)\cos\theta\right] + \varepsilon\left[f(\tau)\lambda\sin^2\theta - r'(\tau)\cos\theta\right] = 0,$$

where

$$F(\tau,\theta) = -\frac{M_z(\tau,\theta)}{A}; \qquad f(\tau) = -\frac{M^{\omega}(\tau)}{A};$$

$$r(\tau) = \frac{C}{A}\,\omega_x(\tau); \qquad \omega_x(\tau) = \omega_x\big|_{t=0} \exp\left[\frac{1}{C}\int_0^{\tau} M_x^{\omega x}(\tau)\,d\tau\right];$$

A is the moment of inertia of the body with respect to the axis Oy (or Oz); C is the moment of inertia of the body with respect to the axis Ox.

In the case when τ does not enter explicitly into equations (1), the problem under consideration is classical (see, for example, [(2)]). The aim of the present work is to compute expressions (asymptotic solutions) suitable for investigating the solutions of system (1) in the presence of dependence on τ over a large time interval $0 \le t \le \tau_0/\varepsilon$.

We shall assume that the functions $F(\tau,\theta)$, $f(\tau)$, and $r(\tau)$ are sufficiently smooth for $0 \le \tau \le \tau_0$ and $0 \le \theta \le 2\pi$. It is also obvious that, with respect to θ, these functions have period 2π. For the solution of this problem, here we ⋯

as before ([4,5]), the method of "model" equations ([3]) will be used. The essence of this method is to express the solution of the equations under consideration through the solution of simpler equations (model equations), which nevertheless possess the main features of the original ones. In the case under consideration we choose, as the model system of equations,

$$\varphi^2(\tau)\,\partial^2\theta_0/\partial\omega^2 - \lambda_0^2 \sin\theta_0\cos\theta_0 + r(\tau)\lambda_0\sin\theta_0 + F(\tau,\theta_0) = 0,$$

$$\lambda_0 \sin^2\theta_0 + r(\tau)\cos\theta_0 = G(\tau), \tag{2}$$

where $\varphi(\tau)$ and $G(\tau)$ are arbitrary functions; the variables τ and ω are independent.

We shall assume that the function $\theta_0(\tau,\omega)$, being determined from these equations, has the following properties:

$$\theta_0(\tau,\omega+T_\omega) = \theta_0(\tau,\omega) + \begin{cases} 0, \\ 2\pi, \end{cases} \qquad \theta_0(\tau,-\omega) = \pm\theta_0(\tau,\omega). \tag{3}$$

At the same time, by virtue of (2), the function $\lambda_0(\tau,\omega)$ will be even in ω and will have, with respect to ω, a period also equal to T_ω.

The general solution $\theta_0(\tau,\omega)$ and $\lambda_0(\tau,\omega)$ of system (2) depends on 4 arbitrary functions $\varphi(\tau)$, $G(\tau)$, $H(\tau)$, and $\Omega(\tau)$, one of which (let it be $\Omega(\tau)$) is additive with respect to ω. But in order that the second of equalities (3) be satisfied, the function $\Omega(\tau)$ must be set equal to zero, so that 3 functions remain arbitrary: $\varphi(\tau)$, $G(\tau)$, and $H(\tau)$.

From the general theorems on the dependence of solutions of differential equations on a parameter it follows that if, in the expressions for $\theta_0(\tau,\omega)$ and $\lambda_0(\tau,\omega)$, one expresses τ and ω through t by means of the relations

$$d\tau/dt = \varepsilon, \qquad d\omega/dt = \varphi(\tau), \tag{4}$$

then the functions of t thus obtained will differ from the solution of the system of equations (1) by quantities of order ε on each time interval of order unity. In order for these functions to be suitable for investigating the solution of system (1) for $0 \le t \le \tau_0/\varepsilon$, it is necessary to determine the arbitrary functions $\varphi(\tau)$, $G(\tau)$, and $H(\tau)$ in a suitable way. To this end we introduce the functions

$$\tilde{\theta}(t) = \theta_0(\tau,\omega) + \varepsilon\theta_1(\tau,\omega); \qquad \tilde{\lambda}(t) = \lambda_0(\tau,\omega) + \varepsilon\lambda_1(\tau,\omega). \tag{5}$$

Here τ and ω are expressed through t by means of relations (4). Let us note that when $0 \le t \le \tau_0/\varepsilon$, then $0 \le \tau \le \tau_0$ and $0 \le \omega < \infty$. We shall try, with the aid of these functions, to satisfy the system of equations (1) up to terms of order ε^2 for $0 \le t \le \tau_0/\varepsilon$. Substituting the functions (5) into equations (1) and using (4), we shall have

$$\begin{gathered}
\varphi^2(\tau)\frac{\partial^2\theta_0}{\partial\omega^2} - \lambda_0^2 \sin\theta_0\cos\theta_0 + r(\tau)\lambda_0\sin\theta_0 + F(\tau,\theta_0) + \\
+\varepsilon\left\{\varphi^2(\tau)\frac{\partial^2\theta_1}{\partial\omega^2} + [r(\tau)\sin\theta_0 - \lambda_0\sin 2\theta_0]\lambda_1 + \right. \\
+[F_\theta(\tau,\theta_0) - \lambda_0^2\cos 2\theta_0 + r(\tau)\lambda_0\cos\theta_0]\theta_1 + 2\varphi(\tau)\frac{\partial^2\theta_0}{\partial\omega\,\partial\tau} \\
\left. +[\varphi'(\tau) + f(\tau)\varphi(\tau)]\frac{\partial\theta_0}{\partial\omega}\right\} + \varepsilon^2\Delta_1\left(\tau, \frac{\partial^{k+l}\theta_i}{\partial\tau^k\partial\omega^l}, \frac{\partial^{k+l}\lambda_i}{\partial\tau^k\partial\omega^l}, \varepsilon\right) = 0; \\
\varphi(\tau)\frac{\partial}{\partial\omega}[\lambda_0\sin^2\theta_0 + r(\tau)\cos\theta_0] + \varepsilon\left\{\varphi(\tau)\frac{\partial}{\partial\omega}[\sin^2\theta_0\,\lambda_1 + \right. \qquad (6) \\
+(\lambda_0\sin 2\theta_0 - r(\tau)\sin\theta_0)\theta_1] + \frac{\partial}{\partial\tau}[\lambda_0\sin^2\theta_0 + r(\tau)\cos\theta_0] - \\
\left. -r'(\tau)\cos\theta_0 + f(\tau)\lambda_0\sin^2\theta_0\right\} + \varepsilon^2\Delta_2\left(\tau, \frac{\partial^{k+l}\theta_i}{\partial\tau^k\partial\omega^l}, \frac{\partial^{k+l}\lambda_i}{\partial\tau^k\partial\omega^l}, \varepsilon\right) = 0
\end{gathered}$$

$$(i = 0, 1; \qquad k, l,\ k + l = 0, 1, 2).$$

Here $F_\theta(\tau, \theta_0)$ denotes the partial derivative of the function $F(\tau, \theta)$ with respect to θ. The coefficients Δ_1 and Δ_2 at ε^2 are functions bounded for bounded values of all their arguments except θ_0, which may be unbounded because these functions have period 2π in θ_0.

The terms of order unity in these equations are compensated completely by virtue of the choice of the model equations. The terms of order ε are compensated by choosing the functions $\theta_1(\tau, \omega)$ and $\lambda_1(\tau, \omega)$. For this purpose they must be determined from the equations

$$\begin{aligned}
&\varphi^2(\tau)\frac{\partial^2\theta_1}{\partial\omega^2} + [r(\tau)\sin\theta_0 - \lambda_0\sin 2\theta_0]\lambda_1 + [F_\theta(\tau, \theta_0) - \lambda_0^2\cos 2\theta_0 \\
&\qquad + r(\tau)\lambda_0\cos\theta_0]\theta_1 = -2\varphi(\tau)\frac{\partial^2\theta_0}{\partial\omega\,\partial\tau} - [\varphi'(\tau) + f(\tau, \theta_0)\varphi(\tau)]\frac{\partial\theta_0}{\partial\omega}; \\
&\lambda_1\sin^2\theta_0 + [\lambda_0\sin 2\theta_0 - r(\tau)\sin\theta_0]\theta_1 \\
&\qquad = \frac{1}{\varphi(\tau)}\left\{-[G'(\tau) + f(\tau)G(\tau)]\omega + [r'(\tau) + f(\tau)r(\tau)]\int_0^\omega \cos\theta_0\,d\omega\right\}.
\end{aligned} \tag{7}$$

In these equations, just as in (2), the variables τ and ω are independent. The homogeneous part of this system of equations has the solution $\theta_{1,0} = \partial\theta_0/\partial\omega$, $\lambda_{1,0} = \partial\lambda_0/\partial\omega$, as is easily verified by differentiating equations (2) with respect to ω. Since the order of system (7) is 2, knowledge of one particular solution of the homogeneous system is sufficient in order to write down the general expressions for the functions $\theta_1(\tau, \omega)$ and $\lambda_1(\tau, \omega)$.

The conditions for determining the arbitrary functions $\varphi(\tau)$, $G(\tau)$, and $H(\tau)$ are formulated by considering, in equations (6), the terms of order ε^2. By arguments no different from those given in ([4],[5]), it can be shown that if, for $0 \le \tau \le \tau_0$, the function $\varphi(\tau) \ge \delta > 0$, the period T_ω in equalities (3) does not depend on τ, and, moreover, $\theta_1(\tau, \omega + T_\omega) = \theta_1(\tau, \omega)$, $\lambda_1(\tau, \omega + T_\omega) = \lambda_1(\tau, \omega)$, then the functions Δ_1 and Δ_2 in equations (6) will be bounded by constants independent of ε for $0 \le t \le \tau_0/\varepsilon$.

It can be shown that the condition of independence of the period T_ω from τ makes it possible to determine one of the three functions $\varphi(\tau)$, $G(\tau)$, and $H(\tau)$. The missing two conditions are obtained by analyzing the expressions for the functions $\theta_1(\tau, \omega)$ and $\lambda_1(\tau, \omega)$, while ensuring periodicity of these functions in ω. These conditions are differential relations and have the form

$$\frac{d}{d\tau}\left[\varphi(\tau)\int_0^{T_\omega/2}\left(\frac{\partial\theta_0}{\partial\omega}\right)^2 d\omega\right] + f(\tau)\left[\varphi(\tau)\int_0^{T_\omega/2}\left(\frac{\partial\theta_0}{\partial\omega}\right)^2 d\omega\right]$$
$$+\frac{1}{\varphi(\tau)}\left\{\left[G'(\tau)+f(\tau)G(\tau)\right]\int_0^{T_\omega/2}\lambda_0\,d\omega - \left[r'(\tau)+f(\tau)r(\tau)\right]\int_0^{T_\omega/2}\lambda_0\cos\theta_0\,d\omega\right\} = 0; \tag{8}$$

$$\left[G'(\tau)+f(\tau)G(\tau)\right]\frac{T_\omega}{2} - \left[r'(\tau)+f(\tau)r(\tau)\right]\int_0^{T_\omega/2}\cos\theta_0\,d\omega = 0.$$

In the case $AM_x^{\omega_x}(\tau) = CM^\omega(\tau)$, conditions (8) are integrable and may be written in the form

$$\varphi(\tau)\int_0^{T_\omega/2}\left(\frac{\partial\theta_0}{\partial\omega}\right)^2 d\omega = D\exp\left[-\int_0^\tau f(\tau)\,d\tau\right];$$
$$G(\tau) = G\big|_{\tau=0}\exp\left[-\int_0^\tau f(\tau)\,d\tau\right], \tag{9}$$

where D is an arbitrary constant.

Thus, we have obtained all the relations necessary for computing the functions $\tilde{\theta}(t)$ and $\tilde{\lambda}(t)$, satisfying the system of equations (1) with accuracy up to ε^2 for $0 \leqslant t \leqslant \tau_0/\varepsilon$.

The terms of order ε in the expressions (5) for $\tilde{\theta}(t)$ and $\tilde{\lambda}(t)$ are small oscillating additions to the principal terms, and they usually need not be taken into account; therefore the formulas for computing the asymptotic solutions may be taken in the form

$$\theta_0(t) = \theta_0(\tau,\omega); \qquad (d\theta/dt)_0 = \varphi(\tau)\,\partial\theta_0/\partial\omega, \tag{10}$$

where $\tau = \varepsilon t$; $\omega = \omega_0 + \int_0^t \varphi(\varepsilon t)\,dt$; ω_0 is an arbitrary constant, and similarly for the functions $\lambda_0(t)$ and $(d\lambda/dt)_0$.

The function $\varphi(\tau)$ (the instantaneous frequency) is related to the instantaneous period of oscillation or rotation of the body by the formula $T(\tau) = T_\omega/\varphi(\tau)$. The difference of the instantaneous frequency $\varphi(\tau)$ from zero is the basic condition for the accuracy of the asymptotic solutions.

Let us indicate the case (the Lagrange case) when

$$F(\tau, \theta) = g(\tau) \sin\theta. \tag{11}$$

The solution of the system of model equations may be sought in the form [1]

$$\cos\theta_0(\tau, \omega) = A(\tau) + B(\tau)\,\mathrm{sn}^2[K(\nu(\tau))\omega, \nu(\tau)]. \tag{12}$$

The relations for determining the functions $A(\tau)$, $B(\tau)$, $\nu(\tau)$, as well as the functions $\varphi(\tau)$, $G(\tau)$, and $H(\tau)$, are obtained by substituting (12) into the model equations (2) and into the periodicity conditions (8) (or (9)).

Let us also indicate the case when the solution of the model equation $\theta_0(\tau, \omega)$ and $\lambda_0(\tau, \omega)$ does not depend on ω. Equations (2) and (8) under this assumption are written in the form

$$-\lambda_0^2 \sin\theta_0 \cos\theta_0 + r(\tau)\lambda_0 \sin\theta_0 + F(\tau, \theta_0) = 0;$$

$$\lambda_0 \sin^2\theta_0 + r(\tau)\cos\theta_0 = G(\tau); \tag{13}$$

$$G' + f(\tau)G = \cos\theta_0\,[r'(\tau) + f(\tau)r(\tau)].$$

The motion is a regular precession with slowly varying parameters $\theta_0(\tau)$, $\lambda_0(\tau)$, and $G(\tau)$; to determine these parameters it is in fact necessary to solve only one first-order differential equation. Note that equations (13) can be obtained directly from system (1), if one assumes that its solutions are functions of $\tau = \varepsilon t$, and, using this, neglects the first and last terms in the first of equations (1), which turn out to be of order ε^2.

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Note: Figure translations are in progress. See original paper for figures.