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Abstract

Full Text

MATHEMATICS

E. B. VINBERG

HOMOGENEOUS CONES

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A **cone** in this note means a domain V in a real finite-dimensional linear space R , invariant under multiplication by positive numbers. By $\mathcal{G}(V)$ we shall denote the group of all linear transformations of the space R leaving the cone V invariant. If $\mathcal{G}(V)$ acts transitively on V , then the cone V is called **homogeneous**. A cone is called **convex** if it contains no straight line and together with any two vectors contains their sum. Let V be a convex cone. Introduce in the space R some Euclidean metric and put

$$V^* = \{a : (a, x) > 0 \text{ for all } x \in \bar{V} \setminus 0\}$$

(\bar{V} denotes the closure of V). The convex cone V^* is called **dual** to V . A convex cone V is called **self-dual** if, for a suitable choice of the Euclidean metric, $V = V^*$.

With each homogeneous cone V there is associated the homogeneous domain $\operatorname{Re} z \in V$ in complex space. In this correspondence, to a convex cone there corresponds a domain analytically equivalent to a bounded one, and to a self-dual cone—a symmetric domain. Therefore the study of convex homogeneous cones is of interest in connection with the problem of classifying bounded homogeneous domains in the space of several complex variables. In this note all self-dual convex homogeneous cones are found and examples of non-self-dual convex homogeneous cones are given. The existence of the latter is reflected in the existence of nonsymmetric bounded homogeneous domains in complex space. Such domains were discovered, contrary to the hypothesis of É. Cartan, by I. I. Pyatetskii-Shapiro ⁽¹⁾. M. Koecher studied ⁽²⁾ a class of convex cones broader than the class of self-dual cones (“nondegenerate domains of positivity”). The following theorem generalizes one of Koecher’ s results to the case of arbitrary convex cones; its proof is the same as Koecher’ s.

Theorem 1. *Let V be a convex cone. The measure with density*

$$\varphi(a) = \int_{V^*} e^{-(a,x)} dx, \quad a \in V,$$

is invariant with respect to all transformations from $\mathcal{G}(V)$. The invariant metric $g = d^2 \ln \varphi$ is positive definite.

Theorem 2. *If V is a convex cone and a closed subgroup \mathcal{G} of the group $\mathcal{G}(V)$ acts transitively on V , then the stationary subgroup $\mathcal{H} \subset \mathcal{G}$ of any point $\xi \in V$ is a maximal compact subgroup in the group \mathcal{G} .*

The set $K = V \cap (\xi - V)$ is bounded and invariant with respect to transformations from \mathcal{H} . Therefore the group \mathcal{H} is compact. Let \mathcal{H} be contained in a compact subgroup $\mathcal{H}_1 \subset \mathcal{G}$. The group \mathcal{H}_1 must have a fixed point in the cone V (for the proof it is enough to apply

all transformations from \mathcal{H}_1 to some ball lying entirely in V , and take the center of gravity of the body so obtained). Consequently, \mathcal{H}_1 is conjugate to some subgroup of the group \mathcal{H} , and therefore $\mathcal{H}_1 = \mathcal{H}$. The theorem is proved.

We shall carry out the classification of convex homogeneous self-dual cones with the aid of the theory of Jordan algebras. A **Jordan algebra** is a commutative algebra A in which the identity

$$[a^2ba] = 0 \tag{1}$$

holds (square brackets denote the associator: $[abc] = a(bc) - (ab)c$). Introduce the notation $T_a x = ax$, $s(a) = \text{Sp } T_a$ ($a, x \in A$). A Jordan algebra is called **semisimple** if the bilinear form $(a, b) = s(ab)$ is nondegenerate. A Jordan algebra over the field of real numbers is called **compact** if $s(a^2) > 0$ for all $a \neq 0$. A semisimple Jordan algebra A decomposes uniquely into a direct sum of simple subalgebras A_i ($i = 1, 2, \dots, k$), and, if the algebra A is compact, then the A_i are compact as well.

In every Jordan algebra the identity (see (3))

$$[[T_a, T_b], T_c] = T_{[acb]}. \tag{2}$$

holds.

Lemma. *Let A be a commutative algebra in which (2) holds. If the bilinear form $(a, b) = s(ab)$ in the algebra A is nondegenerate, then the algebra A is Jordan.*

From (2) it follows, first, that for any $x, y, z \in A$

$$s([xyz]) = 0, \tag{3}$$

and, second, the identity

$$[a^2ba]x = [(a^2b)ax] + [a^2b(ax)] - [a^2(ba)x] + \frac{2}{3}[aa[bax]] - \frac{1}{3}[ba^3x], \tag{4}$$

which is verified by direct calculation. Comparing (3) and (4), we find that for any $a, b, x \in A$

$$s([a^2ba]x) = 0, \tag{5}$$

and since, by assumption, the bilinear form $(a, b) = s(ab)$ is nondegenerate, (1) follows from (5), and the algebra A is Jordan.

We now formulate the main theorem.

If A is a Jordan algebra, then, as follows from (2), the linear span of operators of the form T_a and $[T_a, T_b]$, $a, b \in A$, is a Lie algebra. We shall denote the corresponding linear group by \mathcal{G}_A .

Theorem 3. *Let A be a compact Jordan algebra. The open kernel V_A of the set $A^2 = \{c^2 : c \in A\}$ is a convex self-dual homogeneous cone on which the group \mathcal{G}_A acts transitively. The cone V_A contains the identity e of the algebra A and coincides with the connected component of the set of those $a \in A$ for which the operator T_a is nondegenerate. In the manner described, all convex homogeneous self-dual cones are obtained.*

The proof of the first part of this theorem is not difficult, but requires space, and therefore we shall prove here only its second part.* Let V be a convex homogeneous self-dual cone. From self-duality it follows that the group $\mathcal{G}(V)$, together with every operator P , contains the operator P^* conjugate to it. The same applies to the Lie algebra G of the group $\mathcal{G}(V)$; therefore G decomposes into a direct sum of subspaces:

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$G = H + A$, where H consists of skew-symmetric operators, and A of symmetric operators. Obviously, $[H, H] \subset H$, $[H, A] \subset A$, $[A, A] \subset H$.

The subgroup $\mathcal{H}_0 \subset \mathcal{G}(V)$ generated by the subalgebra H is a maximal connected compact subgroup and therefore coincides with the connected component of the stationary subgroup \mathcal{H} of some point $\xi \in V$. Consequently, H is the Lie algebra of the group \mathcal{H} .

If $a \in A$, let T_a denote the corresponding linear operator. We shall identify the space R , in which the cone V lies, with A in such a way that $a \sim T_a \xi$. In the space A introduce multiplication by the formula $ab = T_{ab} \xi$. Since $T_{ab} - T_{ba} = [T_a, T_b] \xi = 0$, the multiplication introduced by us is commutative. Further,

$$[[T_a, T_b], T_c] = T_{[[a, b], c]} \xi$$

and

$$[[a, b], c] = [[T_a, T_b], T_c] \xi = [T_a, T_b] c = [acb]$$

(the associator). Consequently, the identity (2) holds in the algebra A .

Let φ be the density of the invariant measure in the cone V (see Theorem 1), and let $a \in A$. Taking $\varphi(\xi) = 1$, we have

$$\ln \varphi(\exp tT_a \cdot \xi) = -ts(a),$$

where $s(a) = \text{Sp } T_a$. Expanding the left-hand side in powers of t and taking into account the terms of first and second order, we find that at the point ξ

$$(d^2 \ln \varphi)(a) = s(a^2).$$

The quadratic form $d^2 \ln \varphi$, by Theorem 1, is positive definite; therefore in the algebra A , $s(a^2) > 0$ for $a \neq 0$, and, all the more, the bilinear form $(a, b) = s(ab)$ is nondegenerate. Applying the lemma, we see that the space A with the multiplication introduced by us is a compact Jordan algebra. The point ξ is the identity of the algebra A .

Let us prove that $V = V_A$. The algebra G contains the operators T_a , $a \in A$, and therefore $\mathcal{G}(V) \supset \mathcal{G}_A$, $V \supset V_A$. The density φ of the invariant measure tends to infinity without bound upon approaching the boundary of the cone V_A (see (2)); therefore V_A cannot be a proper subset of V . Hence $V = V_A$, as was to be proved.

Suppose two cones are given: $V_1 \subset R_1$, $V_2 \subset R_2$. The sum $V_1 + V_2$ of the cones V_1 and V_2 is the cone $V \subset R_1 + R_2$ consisting of all pairs (a_1, a_2) with $a_1 \in V_1$, $a_2 \in V_2$.

If A_i ($i = 1, 2, \dots, k$) are compact Jordan algebras and $A = \sum A_i$, then $V_A = \sum V_{A_i}$. Therefore every self-adjoint convex homogeneous cone decomposes into a sum of cones corresponding to simple compact Jordan algebras. Such algebras are not difficult to find from the classification theorems for simple Jordan algebras^(3, 4). These algebras are:

- 1) the algebra of real symmetric matrices of order n ;
- 2) the algebra of complex Hermitian matrices of order n ;
- 3) the algebra of quaternionic Hermitian matrices of order n ;
- 4) the algebra of octonionic Hermitian matrices of order 3;
- 5) the algebra of numbers of the form $c + \sum_1^n c_i \varepsilon_i$, where $\varepsilon_i^2 = 1$, $\varepsilon_i \varepsilon_j = 0$ ($i \neq j$); c, c_i are real.

In cases 1)–4) the Jordan multiplication is defined by the formula

$$ab = \frac{1}{2}(a \cdot b + b \cdot a),$$

where the dot denotes ordinary matrix multiplication.

We now consider a class of homogeneous cones broader (as will be clear from Theorem 4) than the class of convex self-adjoint homogeneous cones, namely the

class of symmetric cones. A cone is called **symmetric** if a linear connection of a symmetric space can be introduced in it in such a way that parallel translations are linear transformations. Obviously every symmetric cone is homogeneous. A linear connection Γ in a cone V is called **nondegenerate** if the tensor $\rho : \rho_{ij} = \partial_i \Gamma_{jk}^k$ (in a linear coordinate system) is nondegenerate. A symmetric cone is called **nondegenerate** if a nondegenerate linear connection can be introduced in it.

of a symmetric space, under which parallel translations are linear transformations. Just as convex self-dual cones are connected with compact Jordan algebras, so nondegenerate symmetric cones turn out to be connected with semisimple Jordan algebras.

An element a of a semisimple Jordan algebra A is called **regular** if the subalgebra generated by a contains the identity e of the algebra A . An element $a^{-1} \in A$ is called the **canonical inverse** to a if $aa^{-1} = e$ and $[T_a, T_{a^{-1}}] = 0$. One can show that every regular element has a unique canonical inverse and that every element having a canonical inverse is regular.

Theorem 4. *Let A be a semisimple Jordan algebra. The connected component V_A containing the identity of the set of regular elements of the algebra A is a homogeneous cone on which the group \mathfrak{G}_A acts transitively. On the cone V_A one can introduce in a unique way a linear connection of a symmetric space, so that parallel translations are contained in the group \mathfrak{G}_A . This connection is nondegenerate; the symmetry σ at the point e is given by the formula $\sigma a = a^{-1}$, where a^{-1} is the canonical inverse to a ; the group \mathfrak{G}_A coincides with the group generated by the parallel translations. In the manner described, all nondegenerate symmetric cones are obtained.*

We omit the proof of this theorem.

In conclusion we give the simplest examples of non-self-dual convex homogeneous cones. Consider the 5-dimensional space R of symmetric matrices $A = (a_{ij})$ of order 3 for which $a_{13} = a_{31} = 0$, and in this space the cone V consisting of all positive definite matrices. It is obvious that the cone V is convex. One can show that it is homogeneous. The dual cone V^* (embedded in the same space R) is given by the inequalities

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} > 0.$$

The cones V and V^* are not isomorphic and, all the more, neither of these cones can be self-dual.

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Moscow State University
named after M. V. Lomonosov

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