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Abstract

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MATHEMATICS

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ON FACTOR-REPRESENTATIONS OF A LOCALLY COMPACT GROUP

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As is known, every unitary representation $g \to U_g$ of a locally compact* group decomposes into a (generally speaking, continuous) direct sum of irreducible representations (see, for example, (1) or (2), Ch. II). In this decomposition there may occur representations that are equivalent to one another. It is natural to pose the question of such a decomposition in which all mutually equivalent representations are collected together, so that the continuous sum is already taken over classes of mutually equivalent representations**.

In the present paper decompositions of a representation into factor-representations are studied; from the results obtained there follows, in particular, a positive solution of the question posed above for groups of type I***.

We note that all the results set forth remain valid for $*$-representations of separable normed rings with involution.

1. Canonical decomposition of a representation into factor-representations. In what follows G will denote a locally compact group satisfying the second axiom of countability, and the term "representation U of the group G" will mean a continuous unitary representation $g \to U_g$ of the group G in a separable Hilbert space $\mathfrak{H}$. A representation U is called a **factor-representation** if the weakly closed ring M, generated by all U_g, $g \in G$, is a factor in the sense of Murray and von Neumann. As is known, if the space $\mathfrak{H}$ of the representation U is decomposed into a continuous sum

$$\mathfrak{H} = \int_\Lambda \mathfrak{H}(\lambda)\, d\mu(\lambda)$$

with respect to the center Z of the ring M^{****}, then the representation U decomposes into representations $U(\lambda)$ in $\mathfrak{H}(\lambda)$, which will be factor-representations for almost all $\lambda \in \Lambda$. This decomposition is called the **canonical decomposition** of the representation into factor-representations.

Theorem 1[*****]. *Let*

$$\mathfrak{H} = \int_{\Lambda} \mathfrak{H}(\lambda)\, d\mu(\lambda)$$

and $U = \{U(\lambda)\}$ *be the canonical decomposition of the representation* U*. Then there exists a set* $\Lambda_0 \subset \Lambda$ *of* μ*-measure zero such that for any* $\lambda, \lambda' \in \Lambda - \Lambda_0$, $\lambda \neq \lambda'$*, the representations* $U(\lambda)$ *and* $U(\lambda')$ *are disjoint*[******].

If G is a group of type I, then $U(\lambda)$ is a factor-representation of type I and therefore is a multiple of an irreducible representation; the disjointness of $U(\lambda)$ and $U(\lambda')$ means then that these representations are multiples of nonequivalent irreducible representations.

The application of Theorem 1 to the case under consideration thus leads to the following theorem, which answers the question posed at the beginning of the article.

Theorem 2. Let

$$\mathfrak{H} = \int_{\Lambda} \mathfrak{H}(\lambda)\, d\mu(\lambda)$$

and $U = \{U(\lambda)\}$ be the canonical decomposition of a representation U of a group G of type I.

Then there exists a set $\Lambda_0 \subset \Lambda$ of μ-measure zero and measurable families $\mathfrak{H}_k(\lambda)$, $k = 1, 2, ...$, such that:

1) for $\lambda, \lambda' \in \Lambda - \Lambda_0$, $\lambda \neq \lambda'$, the representations $U(\lambda)$ and $U(\lambda')$ are multiples of nonequivalent irreducible representations;

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[*] By compactness here and below is meant bicompactness in the sense of P. S. Aleksandrov.

[**] This question also arises in connection with certain problems in probability theory, to which A. M. Yaglom drew the author' s attention.

[***] For the definition of a ring of type I and a group of type I see (2,3); see also (4), Ch. I.

[****] That is, so that Z consists of all operators $A = \{c(\lambda)1\}$, where $c(\lambda)$ is a numerical function in $L^{\infty}_{\mu}(\Lambda)$.

[*****] A special case of this theorem was recently obtained by Guichardet (5).

[******] Two representations U, V are called **disjoint** (see (3), p. 1) if no part of the representation U is equivalent to any part of the representation V.

2) $\mathfrak{H}(\lambda) = \sum_k \mathfrak{H}_k(\lambda)$ for $\lambda \in \Lambda - \Lambda_0$*;
3) $\mathfrak{H}_k(\lambda)$ is invariant with respect to $U(\lambda)$ for $\lambda \in \Lambda_0$;
4) if $\lambda \in \Lambda - \Lambda_0$ and $\mathfrak{H}_k(\lambda) \neq (0)$, then the restriction of $U(\lambda)$ to $\mathfrak{H}_k(\lambda)$ is irreducible.

Remark. The assertion of Theorem 2 will not be valid for arbitrary representations of a group not of type I. Indeed, the assertion of Theorem 2 means that in the canonical decomposition of the representation U almost all the $U(\lambda)$-factors are of type I; hence it follows (see, for example, (^{2}), exercise on p. 125) that U is a representation of type I. Therefore, if G is a group not of type I, then there exist representations of the group G for which the assertion of Theorem 2 will not be valid.

2. Continuous sum of quasi-equivalent factor representations

Theorem 3. Let the representation U in the space $\mathfrak{H}$ be a continuous sum of representations $U(\lambda)$ in the spaces $\mathfrak{H}(\lambda)$, so that

$$\mathfrak{H} = \int_\Lambda \mathfrak{H}(\lambda)\, d\mu(\lambda)$$

and $U = \{U(\lambda)\}$, and let there exist a set $\Lambda' \subset \Lambda$ of μ-measure zero such that all the representations $U(\lambda)$, for $\lambda \in \Lambda - \Lambda'$, are pairwise quasi-equivalent** factor representations. Then U is also a factor representation.

If, in addition, all $U(\lambda)$, for $\lambda \in \Lambda - \Lambda'$, are factor representations of type I, then U is also a factor representation of type I and therefore is a finite or countable discrete sum of mutually equivalent irreducible representations***.

Corollary. Let representations U_1 and U_2 in the spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be continuous sums of representations $U_1(\lambda_1)$, $U_2(\lambda_2)$ in the spaces $\mathfrak{H}_1(\lambda_1)$, $\mathfrak{H}_2(\lambda_2)$, so that

$$\mathfrak{H}_1 = \int_{\Lambda_1} \mathfrak{H}_1(\lambda_1)\, d\mu_1(\lambda_1), \quad U_1 = \{U_1(\lambda_1)\},$$

$$\mathfrak{H}_2 = \int_{\Lambda_2} \mathfrak{H}_2(\lambda_2)\, d\mu_2(\lambda_2), \quad U_2 = \{U_2(\lambda_2)\},$$

and let there exist sets $\Lambda_1' \subset \Lambda_1$, $\Lambda_2' \subset \Lambda_2$ of respectively ν_1- and ν_2-measure zero such that all $U_1(\lambda_1)$, $\lambda_1 \in \Lambda_1 - \Lambda_1'$, and $U_2(\lambda_2)$, $\lambda_2 \in \Lambda_2 - \Lambda_2'$, are factor representations quasi-equivalent to one another. Then U_1 and U_2 are quasi-equivalent factor representations.

3. Application to positive-definite functions

Applying Theorem 2 of § 1 to the representation defined by the given— positive-definite function, we arrive at the following result.

Theorem 4. *Every continuous positive-definite function $\varphi(g)$ on a group G of type I can be represented in the form*

$$\varphi(g) = \int_\Lambda \left[\sum_k \varphi_k(g, \lambda) \right] d\mu(\lambda),$$

where $\varphi_k(g, \lambda)$ are elementary continuous positive-definite functions of g and measurable functions of λ such that:

1) $\varphi_k(g, \lambda)$ and $\varphi_\ell(g, \lambda)$ define equivalent irreducible representations;
2) $\varphi_k(g, \lambda)$ and $\varphi_\ell(g, \lambda')$, for $\lambda \neq \lambda'$, define inequivalent irreducible representations.

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* For some $\lambda \in \Lambda$ it may be that $\mathfrak{H}_k(\lambda) = (0)$.

** Two representations U, V are called **quasi-equivalent** (see (3), § 1) if no part of U is disjoint from V and no part of V is disjoint from U.

*** The first assertion of the theorem is a continuous analogue of a proposition of Mackey (see (3), Lemma 1.2); the second assertion generalizes results of Mautner (6) and Pukanszky (7).

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Note: Figure translations are in progress. See original paper for figures.

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