MATHEMATICS

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Abstract

Full Text

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ON SPACES OF AFFINE CONNECTION ASSOCIATED WITH n VECTOR FIELDS

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1. On an n-dimensional differentiable manifold let us consider n vector fields $\overset{(\alpha)}{a}_i$ $(\alpha,\ i = 1, 2, \ldots, n)$, $\left|\overset{(\alpha)}{a}_i\right| \neq 0$, and let $\overset{(\alpha)}{a}_i \overset{(\beta)}{a}{}^i = \delta^\alpha_\beta$, $\overset{(\alpha)}{a}_i \overset{(\alpha)}{a}{}^j = \delta^j_i$. We introduce a torsion-free affine connection satisfying the conditions $\nabla_k \overset{(\alpha)}{a}_i = \frac{1}{2}\,\partial \overset{(\alpha)}{a}_{[i} / \partial x^{k]}$. Obviously, such a connection is defined uniquely, and

$$\Gamma^k_{ij} = \frac{1}{2}\, \overset{(\alpha)}{a}{}^k \left(\frac{\partial \overset{(\alpha)}{a}_i}{\partial x^j} + \frac{\partial \overset{(\alpha)}{a}_j}{\partial x^i} \right).$$

We shall call it a B-connection, and denote the differentiable manifold on which it is defined by B_n.

As is known, the object of a general torsion-free affine connection possessing n covariantly constant vector fields $\overset{(\alpha)}{a}_i$ has the form

$$G^k_{ij} = \overset{(\alpha)}{a}{}^k\, \frac{\partial \overset{(\alpha)}{a}_j}{\partial x^i}. \tag{1}$$

Thus, $\Gamma^k_{ij} = \frac{1}{2} G^k_{(ij)}$, i.e. the B-connection is the associated torsion-free connection for a flat affine connection.

Theorem 1. *An affine connection is a B-connection if and only if the equations of the geodesics have the form*

$$\frac{dx^i}{dt} = c^\alpha\, \overset{(\alpha)}{\xi}{}^i, \tag{2}$$

where t is an affine parameter; c^α are arbitrary constants; $\overset{(\alpha)}{\xi}{}^i$ are n vector fields.

Substituting (1) into the equations of geodesics

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0,$$

we obtain $c^\alpha c^\beta \overset{(\beta)}{\xi^j} \nabla_j \overset{(\alpha)}{\xi^i} = 0$ for arbitrary c^α, whence $\overset{(\beta)}{\xi^j} \nabla_j \overset{(\alpha)}{\xi^i} + \overset{(\alpha)}{\xi^j} \nabla_j \overset{(\beta)}{\xi^i} = 0$; after transformations we obtain

$$\Gamma^k_{ij} = \frac{1}{2} \overset{(\alpha)}{\xi^k} \frac{\partial \overset{(\alpha)}{\xi}_{(i}}{\partial x^{j)}}, \quad \text{where} \quad \overset{(\alpha)}{\xi_i} \overset{(\beta)}{\xi^i} = \delta^\alpha_\beta, \quad \overset{(\alpha)}{\xi_i} \overset{(\alpha)}{\xi^j} = \delta^j_i;$$

i.e. we have a B-connection. The necessity is easily verified.

Introduce the tensor

$$S^k_{ij} = \frac{1}{2} \overset{(\alpha)}{a^k} \frac{\partial \overset{(\alpha)}{a}_{[i}}{\partial x^{j]}}.$$

One can verify that the identity

$$\nabla_{(i} S^l_{jk)} - S^l_{p(i} S^p_{jk)} = 0 \tag{3}$$

holds. Taking into account that $\Gamma^k_{ij} = G^k_{ij} - S^k_{ij}$, we easily obtain that $R_{ij,k}{}^l = \nabla_{[i} S^l_{j]k} - S^l_{m[i} S^m_{j]k}$, or, in view of (3),

$$R_{ij,k}{}^l = S^l_{mk} S^m_{ij} - \nabla_k S^l_{ij}. \tag{4}$$

2. A space B_n with the property $\nabla_k a = 0$ ($a = \det \|a^{(\alpha)}_i\|$) will be called a **space with invariant volume**.

Let us introduce into consideration the vector $S_i = S^k_{ik}$.

Theorem 2. *The space B_n is equiaffine if and only if $\nabla_{[i} S_{j]} = 0$, and possesses invariant volume if and only if $S_i = 0$.*

Indeed, $R_{jk} = \nabla_k S_j - S^l_{mk} S^m_{jl}$, $R_{[jk]} = \nabla_{[k} S_{j]}$, whence the first assertion follows. Further, we have

$$\nabla_k a = \frac{1}{2} a \left(\frac{\partial \ln a}{\partial x^k} - a^l_{(\alpha)} \frac{\partial a^{(\alpha)}_k}{\partial x^l} \right);$$

but

$$S_k = \frac{1}{2} \left(\frac{\partial \ln a}{\partial x^k} - a^i_{(\alpha)} \frac{\partial a^{(\alpha)}_k}{\partial x^i} \right) = \frac{\nabla_k a}{a}.$$

Thus every space with invariant volume is equiaffine.

We shall call the space $\{a_i^{(\alpha)}\}$ **conformal** to the space $\{\bar{a}_i^{(\alpha)}\}$, if

$$\bar{a}_i^{(\alpha)} = e^{\sigma} a_i^{(\alpha)};$$

if $\sigma = \mathrm{const}$, then the spaces will be called **similar**. It is easy to see that a conformal mapping preserves equiaffineness.

Theorem 3. *Among all mutually conformal equiaffine spaces B_n there is exactly one space with invariant volume (up to similarity).*

Indeed,

$$\bar{S}_{ij}^k = S_{ij}^k + \frac{1}{2}\delta_{(i}^k \sigma_{j)},$$

therefore

$$\bar{S}_i = S_i + \frac{1}{2}(n-1)\sigma_i$$

($\sigma_i = \partial\sigma/\partial x^i$), whence the assertion of the theorem follows.

3. The collection $\{a_i^{(\alpha)}\}$ of vectors determining the space B_n will be called the **fundamental coreper** of the space; correspondingly $\{a_{(\alpha)}^i\}$ will be called the **fundamental reper**. A fundamental coreper consisting of gradient vectors can be reduced to the form $a_i^{(\alpha)} = \delta_i^\alpha$; the corresponding object $\Gamma_{ij}^k = 0$, and the space is flat. But not only a gradient coreper determines a flat space. Put

$$\Gamma_{ij}^k \equiv \frac{1}{2} a_{(\alpha)}^k \left(\frac{\partial a_i^{(\alpha)}}{\partial x^j} + \frac{\partial a_j^{(\alpha)}}{\partial x^i} \right) = 0;$$

then

$$\frac{\partial a_i^{(\alpha)}}{\partial x^j} + \frac{\partial a_j^{(\alpha)}}{\partial x^i} = 0,$$

whence

$$\frac{\partial^2 a_i^{(\alpha)}}{\partial x^j \partial x^k} + \frac{\partial^2 a_j^{(\alpha)}}{\partial x^i \partial x^k} = 0,$$

therefore

$$\frac{\partial^2 a_j^{(\alpha)}}{\partial x^k \partial x^i} + \frac{\partial^2 a_k^{(\alpha)}}{\partial x^j \partial x^i} = 0,$$

$$\frac{\partial^2 a_k^{(\alpha)}}{\partial x^i \partial x^j} + \frac{\partial^2 a_i^{(\alpha)}}{\partial x^k \partial x^j} = 0.$$

Subtracting the 2nd equality from the sum of the 1st and 3rd, we obtain

$$\frac{\partial^2 a_i^{(\alpha)}}{\partial x^j \partial x^k} = 0,$$

whence

$$a_i^{(\alpha)} = c_{ij}^{\alpha} x^j + c_i^{\alpha}.$$

Theorem 4. *Every flat B_n can be given in the form*

$$a_i^{(\alpha)} = c_{ij}^{\alpha} x^j + c_i^{\alpha}, \tag{5}$$

where c_i^{α}, $c_{ij}^{\alpha} = -c_{ji}^{\alpha}$ are constants.

4. In view of the fact that $\Gamma_{ij}^k = \frac{1}{2} G_{(ij)}^k$, and that every flat connection can be reduced to the form $G_{ij}^k = \delta_j^k \, \partial \ln \varphi_j / \partial x^i$ (1), we obtain that every B-connection can be reduced to the form

$$\Gamma_{ij}^k = \frac{1}{2} \left(\delta_i^k \frac{\partial \ln \varphi_i}{\partial x^j} + \delta_j^k \frac{\partial \ln \varphi_j}{\partial x^i} \right).$$

The curvature tensor in this coordinate system has the form

$$R_{lk,i}{}^{q} = \delta_i^q A_{lik} - \delta_k^q A_{kil},$$

where

$$A_{lik} = \frac{1}{4} \left(2 \frac{\partial^2 \ln \varphi_l}{\partial x^i \partial x^k} + \frac{\partial \ln \varphi_l}{\partial x_k} \frac{\partial \ln \varphi_l}{\partial x_l} - \frac{\partial \ln \varphi_l}{\partial x^k} \frac{\partial \ln \varphi_k}{\partial x^i} - \frac{\partial \ln \varphi_l}{\partial x^i} \frac{\partial \ln \varphi_i}{\partial x^k} \right).$$

If the space is projectively flat and $n > 2$, then $R_{lk,i}{}^{q} = \delta_{[k}^q P_{l]i} + \delta_i^q P_{[lk]}$, where $P_{ki} = -(nR_{ki} + R_{ik})/(n^2 - 1)$. Let $q = i \neq k, l$. Then $P_{[lk]} = 0$, i.e. $R_{[lk]} = 0$, and we obtain that for $n > 2$ every projectively flat B_n is equiaffine. For $n = 2$ this can also be proved. But every equiaffine projectively flat space can be given in the form $\Gamma^k_{ij} = \frac{1}{2}\delta^k_{(i}\tau_{j)}$, where $\tau_j = \partial\tau/\partial x^j$, i.e. $\partial \overset{(\alpha)}{a}_i/\partial x^j + \partial \overset{(\alpha)}{a}_j/\partial x^i = \overset{(\alpha)}{a}_i\tau_j + \overset{(\alpha)}{a}_j\tau_i$; introducing $\overset{(\alpha)}{A}_i = e^{-\tau}\overset{(\alpha)}{a}_i$, we obtain $\partial \overset{(\alpha)}{A}_{(i}/\partial x^{j)} = 0$, whence $\overset{(\alpha)}{A}_i = c^\alpha_{ij}x^j + c^\alpha_i$, where $c^\alpha_{(ij)} = 0$. Thus:

Theorem 5. *The fundamental coframe of any projectively flat B_n can be represented in the form*

$$\overset{(\alpha)}{a}_i = e^\tau(c^\alpha_{ij}x^j + c^\alpha_i), \tag{6}$$

where $c^\alpha_{ij} = -c^\alpha_{ji}$, c^α_i are constants; i.e. every projectively flat B_n is conformally flat, and conversely.

Corollary. Every equiprojective connection admits a fundamental coframe, i.e. is a B-connection.

Indeed, the equiprojective connection $\Gamma^k_{ij} = \frac{1}{2}\delta^k_{(i}\tau_{j)}$ admits the coframe (6) with the property $\nabla^k \overset{(\alpha)}{a}_i = \frac{1}{2}\partial \overset{(\alpha)}{a}_{[i}/\partial x^{k]}$.

We shall call a space with a fundamental coframe of the form $\overset{(\alpha)}{a}_i = e^\tau \overset{(\alpha)}{A}_i$, where $\overset{(\alpha)}{A}_i = \partial \overset{(\alpha)}{A}/\partial x^i$, **conformally gradient**. Obviously, this is a special case of projectively flat B_n. It is easy to see that for a conformally gradient B_n

$$\frac{\partial \overset{(\alpha)}{a}_{[i}}{\partial x^{j]}} = \tau_{[j}\overset{(\alpha)}{a}_{i]}, \tag{7}$$

whence, contracting with a^j, we obtain $\tau_j = \dfrac{2}{n-1}S_j$, and therefore

$$S^k_{ij} = \frac{1}{n-1}S_{[i}\delta^k_{j]}. \tag{8}$$

Conversely, if conditions (8) are satisfied, and S_i is a gradient, then, putting $\tau_i = \dfrac{2}{n-1}S_i$, we obtain (7), whence, representing $\overset{(\alpha)}{a}_i$ in the form $e^\tau \overset{(\alpha)}{A}_i$, we obtain $\partial \overset{(\alpha)}{A}_{[i}/\partial x^{j]} = 0$. Thus the following is true:

Theorem 6. *The space B_n is conformally gradient if and only if S_i is a gradient and*

$$S^k_{ij} = \frac{1}{n-1}S_{[i}\delta^k_{j]}.$$

5. It is easy to see that spaces without torsion, defined by a simply transitive group (see (2); we shall call such spaces group spaces), form a subclass of the spaces B_n; namely, the group spaces are those B_n for which the vectors $\overset{(\alpha)}{a}{}^i$

of the fundamental frame are generators of a certain simply transitive group, i.e. satisfy the conditions

$$\overset{(\alpha)}{a}{}^k \frac{\partial \overset{(\beta)}{a}{}^i}{\partial x^k} - \overset{(\beta)}{a}{}^k \frac{\partial \overset{(\alpha)}{a}{}^i}{\partial x^k} = c^{\gamma}_{\alpha\beta} \overset{(\gamma)}{a}{}^i, \tag{9}$$

where $c^{\gamma}_{\alpha\beta}$ are constants, $c^{\gamma}_{(\alpha\beta)} = c^{\delta}_{\varepsilon(\alpha} c^{\varepsilon}_{\beta\gamma)} = 0$.

Transforming (9), we obtain $2a_k \overset{(\beta)}{a}{}^i \overset{(\alpha)}{a}{}^j S^k_{ij} = c^{\gamma}_{\alpha\beta}$. Covariant differentiation of the scalars $c^{\gamma}_{\alpha\beta}$ gives

$$\nabla_k S^l_{ij} - S^l_{m(k} S^m_{ij)} = 0, \tag{10}$$

whence, cycling, we obtain $\nabla_{(k} S^l_{ij)} - 3S^l_{m(k} S^m_{ij)} = 0$; comparing with (3), we have

$$S^l_{m(k} S^m_{ij)} = 0, \tag{11}$$

whence, by virtue of (10),

$$\nabla_k S^l_{ij} = 0. \tag{12}$$

The conditions $c^{\gamma}_{(\alpha\beta)} = 0$ are satisfied, and the Jacobi identities lead to (11). Let us note that, owing to the identities (3), conditions (11) follow from (12). Conversely, if (11) holds, then, as is easy to see, the $c^{\gamma}_{\alpha\beta}$ determined by the system (9) are constant, skew-symmetric, and satisfy the Jacobi identities. Thus, we have proved:

Theorem 7. *In order that the space B_n be a group space, it is necessary and sufficient that conditions (12) hold.*

From Theorem 7 and (4) there immediately follows the known fact: all group spaces are symmetric. Let us note that all these spaces are equiaffine, since their Ricci tensor $R_{ij} = S^l_{kj} S^k_{li}$ is symmetric.

6. By a **motion** we shall mean a transformation of the space B_n preserving the fundamental coframe, i.e., one for which $D_L a^{(\alpha)}_i = 0$ (D_L is the Lie derivative), or

$$\frac{\partial \xi^k}{\partial x^i} a^{(\alpha)}_k + \frac{\partial a^{(\alpha)}_i}{\partial x^k} \xi^k = 0, \tag{13}$$

or, what is the same,

$$\nabla_i \xi^k = S^k_{ij} \xi^j. \tag{14}$$

The integrability conditions for this system are $(\nabla_k S^l_{ij} - S^l_{m(k} S^m_{ij)})\xi^k = 0$, whence:

Theorem 8. *Among the spaces B_n, the group spaces and only they admit an n-parameter group of motions.*

Thus, group spaces are characterized by maximal mobility,

Theorem 9. *All motions of spaces B_n are translations.*

Indeed, the motion $x^{i'} = x^i + \xi^i \delta t$ is a translation, i.e. its trajectories are geodesics, if and only if

$$\xi^k(\xi^l \nabla_k \xi^i - \xi^i \nabla_k \xi^l) = 0,$$

as follows from the equations of geodesics. But for B_n this condition, by virtue of (14), is fulfilled, since

$$\xi^k \nabla_k \xi^i = S^i_{kl} \xi^k \xi^l = 0.$$

Let us consider, in conclusion, motions of conformally gradient spaces $a^{(\alpha)}_i = \delta^\alpha_i e^\sigma$. From (13)

$$\partial \xi^l / \partial x^i + \delta^l_i \sigma_k \xi^k = 0,$$

whence, as is easy to see, $\xi^l = cx^l + c^l$, where the constants c, c^l must satisfy the equation

$$\sigma_k(cx^k + c^k) + c = 0. \tag{15}$$

The number of solutions (c, c^k) of this equation gives the order of the group of motions.

If we now consider the conformally gradient space $A^{(\alpha)}_i = \delta^\alpha_i e^{2\sigma}$, then for it $\Gamma^k_{ij} = \delta^k_{(i} \sigma_{j)}$, and for such a space the number of solutions of equation (15) means the number of parallel fields of contravariant vectors (see ([3])). Thus, we have proved:

Theorem 10. *The groups of motions of conformally gradient spaces are subgroups of the affine group. Their order is equal to the number of parallel fields of contravariant vectors admitted by the space of the same type, but with the conformality factor squared.*

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Note: Figure translations are in progress. See original paper for figures.