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**Abstract**

**Full Text**

## Reports of the Academy of Sciences of the USSR

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**MATHEMATICS**

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### $\varepsilon$ -ENTROPY OF THE CLASSES $E_s^{\alpha,k}(B)$ AND $W_s^\alpha(B)$ IN THE METRIC $L_2$

*(Presented by Academician A. N. Kolmogorov on 16 XI 1959)*

An  $\varepsilon$ -net for a set  $G$  in a normed space  $R$  is a subset  $G_\varepsilon$  of elements of  $R$  such that for every  $f \in G$  there exists  $\tilde{f} \in G_\varepsilon$  such that  $\|f - \tilde{f}\| \leq \varepsilon$  <sup>(1)</sup>. Suppose  $G$  has a finite  $\varepsilon$ -net for every  $\varepsilon > 0$ . If  $N_\varepsilon$  is the smallest number of elements in an  $\varepsilon$ -net for  $G$ , then  $\lg_2 N_\varepsilon = H_\varepsilon(G)$  is called the  $\varepsilon$ -entropy of the set  $G$  relative to  $R$ .

Denote by  $E_s^{\alpha,k}(B)$  for  $\alpha > 1/2$  ( $W_s^\alpha(B)$  for  $\alpha > 0$ ) the class of complex-valued functions  $f(x_1, \dots, x_s)$  having period 1 in each variable and expandable in a Fourier series convergent in the mean,

$$f(x_1, \dots, x_s) = \sum_{m_1 \dots m_s = -\infty}^{\infty} C_{m_1 \dots m_s} \exp[2\pi i(m_1 x_1 + \dots + m_s x_s)],$$

where

$$|C_{m_1 \dots m_s}| \leq B \frac{\ln^k(\bar{m}_1 \dots \bar{m}_s) + 1}{(\bar{m}_1 \dots \bar{m}_s)^\alpha},$$

respectively

$$\left( \sum_{m_1 \dots m_s = -\infty}^{\infty} |(\bar{m}_1 \dots \bar{m}_s)^\alpha C_{m_1 \dots m_s}|^2 = \|f\|_{W_s^\alpha}^2 \leq B^2 \right),$$

where  $\bar{m} = |m|$ , if  $m \neq 0$ , and  $\bar{0} = 1$ . (Concerning the classes  $E_s^{\alpha,k}(B)$ , see <sup>(2,3)</sup>, where the case  $k = 0$  and in fact any  $k < -\alpha s$  is considered.)

It can be shown that  $W_s^\alpha(B)$ , for integral  $\alpha$ , is the set of complex-valued functions  $f(x_1, \dots, x_s)$  having period 1 in each variable and possessing square-summable partial derivatives

$$\partial^{\alpha_1 + \dots + \alpha_s} f / \partial x_1^{\alpha_1} \dots \partial x_s^{\alpha_s}, \quad 0 \leq \alpha_i \leq \alpha.$$

Below the  $\varepsilon$ -entropy of the classes  $E_s^{\alpha, k}(B)$  and  $W_s^\alpha(B)$  relative to  $L_2$  is estimated. For the estimates the method described in <sup>(1)</sup> is used. To describe the limiting behavior of functions, besides the symbol  $O$ , the notations  $f \sim g$ , if  $\lim f/g = 1$ ;  $f \lesssim g$  or  $g \gtrsim f$ , if  $\lim f/g \leq 1$ , are used.

**Theorem 1.**

$$\begin{aligned} & \frac{2^{s+1}}{(s-1)! \ln 2} \frac{1}{\alpha^{s-2}} \left(\frac{B}{\varepsilon}\right)^{1/\alpha} \ln^{s-1} \left(\frac{B}{\varepsilon}\right) \lesssim \\ & \lesssim H_\varepsilon(W_s^\alpha(B)) \lesssim \frac{2^{s+1}}{(s-1)! \ln 2} \frac{(3\sqrt{2})^{1/\alpha}}{\alpha^{s-2}} \left(\frac{B}{\varepsilon}\right)^{1/\alpha} \ln^{s-1} \left(\frac{B}{\varepsilon}\right). \end{aligned} \quad (1)$$

**Proof.** If there is an  $\varepsilon$ -net for  $W_s^\alpha(B)$  consisting of  $N$  elements, then there exist  $N$  points

$$f_i = (\dots, C_{m_1 \dots m_s}^{(i)}, \dots) \quad (i = 1, 2, \dots, N)$$

such that  $N$  “balls”  $\|f - f_i\|_{L_2} \leq \varepsilon$  cover the “ellipsoid”  $\|f\|_{W_s^\alpha} \leq B$ .

In particular, for any  $Q$ ,  $N$  balls

$$\sum_{\bar{m}_1 \dots \bar{m}_s \leq Q} \left| C_{m_1 \dots m_s} - C_{\bar{m}_1 \dots \bar{m}_s}^{(i)} \right|^2 \leq \varepsilon^2 \quad (2)$$

cover the ellipsoid

$$\sum_{\bar{m}_1 \dots \bar{m}_s \leq Q} \left| (\bar{m}_1 \dots \bar{m}_s)^\alpha C_{m_1 \dots m_s} \right|^2 \leq B^2; \quad (3)$$

therefore  $N \geq V/v$ , where  $V$  is the volume of (3), and  $v$  is the volume of (2). Using the formula for the volume of an  $n$ -dimensional ellipsoid, we obtain

$$N \geq \prod_{\bar{m}_1 \dots \bar{m}_s \leq Q} \frac{B^2}{\varepsilon^2 (\bar{m}_1 \dots \bar{m}_s)^{2\alpha}} = \left(\frac{B}{\varepsilon Q^\alpha}\right)^{\frac{2\sum 1}{\bar{m}_1 \dots \bar{m}_s \leq Q}} \prod_{\bar{m}_1 \dots \bar{m}_s \leq Q} \left(\frac{Q}{\bar{m}_1 \dots \bar{m}_s}\right)^{2\alpha}.$$

The first of inequalities (1) will be obtained if we take  $Q = (B/\varepsilon)^{1/\alpha}$  and use

$$\sum_{\bar{m}_1 \dots \bar{m}_s \leq Q} 1 \sim \sum_{\bar{m}_1 \dots \bar{m}_s \leq Q} \ln \left( \frac{Q}{\bar{m}_1 \dots \bar{m}_s} \right) \sim \frac{2^s}{(s-1)!} Q \ln^{s-1} Q. \quad (4)$$

To obtain the second of inequalities (1), one must construct a suitable  $\varepsilon$ -net for  $W_s^\alpha(B)$ . Let  $N_0$  be the maximal number of nonintersecting balls of radius  $\varepsilon/2\sqrt{2}$  with centers in the ellipsoid (3). We shall prove that, for  $Q = (B\sqrt{2}/\varepsilon)^{1/\alpha}$ , the centers of the latter form an  $\varepsilon$ -net for  $W_s^\alpha(B)$ . Indeed, let  $f_0 \in W_s^\alpha(B)$ ,  $f_0 = (\dots, C_{\bar{m}_1 \dots \bar{m}_s}^{(0)}, \dots)$ . Since  $f_0$  satisfies (3), the ball  $\|f - f_0\|_{L_2} \leq \varepsilon$  must intersect one of our  $N_0$  balls; then the distance between their centers does not exceed  $\varepsilon/\sqrt{2}$ , i.e., for some  $k$ ,  $1 \leq k \leq N_0$ ,

$$\sum_{\bar{m}_1 \dots \bar{m}_s \leq Q} |C_{\bar{m}_1 \dots \bar{m}_s}^{(0)} - C_{\bar{m}_1 \dots \bar{m}_s}^{(k)}|^2 \leq \frac{\varepsilon^2}{2},$$

therefore,

$$\begin{aligned} & \sum_{\bar{m}_1 \dots \bar{m}_s \leq Q} |C_{\bar{m}_1 \dots \bar{m}_s}^{(0)} - C_{\bar{m}_1 \dots \bar{m}_s}^{(k)}|^2 + \sum_{\bar{m}_1 \dots \bar{m}_s > Q} |C_{\bar{m}_1 \dots \bar{m}_s}^{(0)}|^2 \leq \\ & \leq \frac{\varepsilon^2}{2} + \sum_{\bar{m}_1 \dots \bar{m}_s > Q} \frac{1}{Q^{2\alpha}} |(\bar{m}_1 \dots \bar{m}_s)^\alpha C_{\bar{m}_1 \dots \bar{m}_s}^{(0)}|^2 \leq \frac{\varepsilon^2}{2} + \frac{B^2}{Q^{2\alpha}} = \varepsilon^2, \end{aligned}$$

i.e., the points with zero coordinates for  $\bar{m}_1 \dots \bar{m}_s > Q$  and with coordinates  $C_{\bar{m}_1 \dots \bar{m}_s}^{(k)}$  for  $\bar{m}_1 \dots \bar{m}_s \leq Q$  form an  $\varepsilon$ -net for  $W_s^\alpha(B)$ . The number of elements of this  $\varepsilon$ -net is equal to  $N_0$ . If a point  $\varphi$  lies in one of our  $N_0$  balls, then it can be represented in the form  $\varphi = \varphi_1 + \varphi_2$ , where  $\varphi_1 \in (3)$ ,  $\|\varphi_2\|_{L_2} \leq \varepsilon/2\sqrt{2}$ . Then, if  $\varphi_2 = (\dots, C_{\bar{m}_1 \dots \bar{m}_s}, \dots)$ , then

$$\begin{aligned} \|\varphi\|_{W_s^\alpha} & \leq \|\varphi_1\|_{W_s^\alpha} + \|\varphi_2\|_{W_s^\alpha} \leq B + \left( \sum_{\bar{m}_1 \dots \bar{m}_s \leq Q} |(\bar{m}_1 \dots \bar{m}_s)^\alpha C_{\bar{m}_1 \dots \bar{m}_s}|^2 \right)^{1/2} \leq \\ & \leq B + Q^\alpha \left( \sum_{\bar{m}_1 \dots \bar{m}_s \leq Q} |C_{\bar{m}_1 \dots \bar{m}_s}|^2 \right)^{1/2} \leq B + Q^\alpha \cdot \frac{\varepsilon}{2\sqrt{2}} = \frac{3B}{2}. \end{aligned}$$

Consequently, all our  $N_0$  balls lie inside the “ellipsoid”  $\|\varphi\|_{W_s^\alpha} \leq 3B/2$ ,

therefore  $N_0 \leq V_1/v_1$ , where  $V_1$  is the volume of this “ellipsoid,” and  $v_1$  is the volume of a ball of the same dimension of radius  $\varepsilon/2\sqrt{2}$ , i.e.

$$N \leq \prod_{\bar{m}_1 \dots \bar{m}_s < Q} \left( \frac{3B/2}{\frac{\varepsilon}{2\sqrt{2}} (\bar{m}_1 \dots \bar{m}_s)^\alpha} \right)^2 = \left( \frac{3\sqrt{2}B}{\varepsilon Q^\alpha} \right)^{\sum_{\bar{m}_1 \dots \bar{m}_s \leq Q} 1} \prod_{\bar{m}_1 \dots \bar{m}_s \leq Q} \left( \frac{Q}{\bar{m}_1 \dots \bar{m}_s} \right)^{2\alpha}.$$

Using (4) and recalling that  $Q = (B\sqrt{2}/\varepsilon)^{1/\alpha}$ , we obtain the second of inequalities (1). The theorem is proved. For the classes  $E_s^{\alpha,k}(B)$  an analogous theorem holds.

**Theorem 2.**

$$H_\varepsilon(E_s^{\alpha,k}(B)) \gtrsim \frac{2}{\ln 2} \left( \frac{2^s}{(s-1)!} \right)^{\frac{2\alpha}{2\alpha-1}} \left( \frac{1}{\alpha-1/2} \right)^{\frac{2k+2\alpha(s-1)}{2\alpha-1}-1} \left( \frac{B}{\varepsilon} \right)^{\frac{1}{\alpha-1/2}} \ln^{\frac{2k+2\alpha(s-1)}{2\alpha-1}} \left( \frac{B}{\varepsilon} \right),$$

$$H_\varepsilon(E_s^{\alpha,k}(B)) \lesssim \frac{2\mu}{\ln 2} \left( \frac{2^s}{(s-1)!} \right)^{\frac{2\alpha}{2\alpha-1}} \left( \frac{1}{\alpha-1/2} \right)^{\frac{2k+2\alpha(s-1)}{2\alpha-1}} \left( \frac{B}{\varepsilon} \right)^{\frac{1}{\alpha-1/2}} \ln^{\frac{2k+2\alpha(s-1)}{2\alpha-1}} \left( \frac{B}{\varepsilon} \right), \quad (5)$$

where

$$\mu = \left( 4.27 \left( 1 + \sqrt[3]{\frac{4}{2\alpha-1}} \right)^3 \right)^{\frac{1}{2\alpha-1}}.$$

**Proof.** The proof of the first of inequalities (5) is carried out analogously to the proof of the first of inequalities (1), with the only difference that instead of the ellipsoid (3) one must consider the “parallelepiped”

$$|C_{m_1 \dots m_s}| \leq B \frac{\ln^k(\bar{m}_1 \dots \bar{m}_s) + 1}{(\bar{m}_1 \dots \bar{m}_s)^\alpha}, \quad \bar{m}_1 \dots \bar{m}_s \leq Q, \quad (6)$$

the ratio of whose volume to the volume of the ball (2) will be

$$\frac{V}{v} = \left( \sum_{\bar{m}_1 \dots \bar{m}_s \leq Q} 1 \right)! \prod_{\bar{m}_1 \dots \bar{m}_s \leq Q} \left( \frac{B \ln^k(\bar{m}_1 \dots \bar{m}_s) + 1}{\varepsilon (\bar{m}_1 \dots \bar{m}_s)^\alpha} \right)^2.$$

Using (4), we obtain

$$N \gtrsim \exp \left\{ \frac{2^s}{(s-1)!} Q \ln^{s-1} Q \left( 2\alpha - 1 + \ln \frac{B^2}{\varepsilon^2} - \frac{\ln^{2k+s-1} Q}{Q^{2\alpha-1}} \frac{2^s}{(s-1)!} \right) \right\}.$$

Taking

$$\frac{Q^{2\alpha-1}}{\ln^{2k+s-1} Q} = \frac{2^s}{(s-1)!} \frac{B^2}{\varepsilon^2},$$

we obtain the first of inequalities (5).

To obtain the second of inequalities (5), we shall prove that the set of functions

$$f_{(k)}(x_1, \dots, x_s) = \sum_{\bar{m}_1 \dots \bar{m}_s \leq Q} \delta \left( k'_{m_1 \dots m_s} + ik''_{m_1 \dots m_s} \right) \exp[2\pi i(m_1 x_1 + \dots + m_s x_s)],$$

where  $k_{m_1 \dots m_s} = k'_{m_1 \dots m_s} + ik''_{m_1 \dots m_s}$  are integer complex numbers in the disks

$$|k_{m_1 \dots m_s}| \leq \frac{B \ln^k(\bar{m}_1 \dots \bar{m}_s) + 1}{\delta} + \frac{\sqrt{2}}{2}, \quad (7)$$

for suitable  $\delta$  and  $Q$ , forms an  $\varepsilon$ -net for  $E_s^{\alpha, k}(B)$ . Indeed, if  $|C| \leq B$ , then there exist integers  $k'$  and  $k''$  such that  $|k' + ik''| \leq B + \sqrt{2}/2$ ,  $|C - (k' + ik'')| \leq \sqrt{2}/2$ . Denote  $k' + ik'' = [C]$ . Let  $f \in E_s^{\alpha, k}(B)$ ,

$$f(x_1, \dots, x_s) = \sum_{m_1 \dots m_s = -\infty}^{\infty} C_{m_1 \dots m_s} \exp[2\pi i(m_1 x_1 + \dots + m_s x_s)],$$

$$\tilde{f}(x_1, \dots, x_s) = \sum_{\bar{m}_1 \dots \bar{m}_s \leq Q} \delta \left[ \frac{C_{\bar{m}_1 \dots \bar{m}_s}}{\delta} \right] \exp[2\pi i(m_1 x_1 + \dots + m_s x_s)].$$

Then  $f$  is one of the functions  $f_{(k)}(x_1, \dots, x_s)$ , and

$$\begin{aligned} \|f - \tilde{f}\|_{L_2^s}^2 &= \sum_{\bar{m}_1 \dots \bar{m}_s \leq Q} \left| C_{\bar{m}_1 \dots \bar{m}_s} - \delta \left[ \frac{C_{\bar{m}_1 \dots \bar{m}_s}}{\delta} \right] \right|^2 + \sum_{\bar{m}_1 \dots \bar{m}_s > Q} |C_{\bar{m}_1 \dots \bar{m}_s}|^2 \leq \\ &\leq \frac{\delta^2}{2} \sum_{\bar{m}_1 \dots \bar{m}_s \leq Q} 1 + B^2 \sum_{\bar{m}_1 \dots \bar{m}_s > Q} \left( \frac{\ln^k(\bar{m}_1 \dots \bar{m}_s) + 1}{(\bar{m}_1 \dots \bar{m}_s)^\alpha} \right)^2. \end{aligned}$$

If  $0 < \lambda < 1$  and

$$B^2 \sum_{\bar{m}_1 \dots \bar{m}_s > Q} \left( \frac{\ln^k(\bar{m}_1 \dots \bar{m}_s) + 1}{(\bar{m}_1 \dots \bar{m}_s)^\alpha} \right)^2 = \lambda \varepsilon^2, \quad \frac{\delta^2}{2} \sum_{\bar{m}_1 \dots \bar{m}_s < Q} 1 = (1 - \lambda) \varepsilon^2, \quad (8)$$

then  $\|f - \tilde{f}\|_{L_2} \leq \varepsilon$ , i.e.  $f_{(k)}(x_1, \dots, x_s)$  indeed form an  $\varepsilon$ -net for  $E_s^{\alpha, k}(B)$ . The number of elements in this  $\varepsilon$ -net is

$$N \leq \prod_{\bar{m}_1 \dots \bar{m}_s < Q} \pi \left( \frac{B \ln^k(\bar{m}_1 \dots \bar{m}_s) + 1}{\delta (\bar{m}_1 \dots \bar{m}_s)^\alpha} + \sqrt{2} \right)^2 \leq \left( \pi \left( \sqrt{2} + \frac{B \ln^k Q}{\delta Q^\alpha} \right)^2 \right)^{\sum 1} \prod_{\bar{m}_1 \dots \bar{m}_s < Q} \left( \frac{Q}{\bar{m}_1 \dots \bar{m}_s} \right)^{2\alpha} \left( \frac{\ln^k(\bar{m}_1 \dots \bar{m}_s) + 1}{\ln^k Q} \right)^2,$$

for the  $k_{\bar{m}_1 \dots \bar{m}_s}$  independently run through all integer points in the circles (7), and the number of integer points in a circle of radius  $R$  does not exceed the area of the circle of radius  $R + \sqrt{2}/2$ . Using (9), (4), and

$$\sum_{\bar{m}_1 \dots \bar{m}_s > Q} \left( \frac{\ln^k(\bar{m}_1 \dots \bar{m}_s) + 1}{(\bar{m}_1 \dots \bar{m}_s)^\alpha} \right)^2 \sim \frac{2^s}{(s-1)!(2\alpha-1)} \frac{\ln^{2k+s-1} Q}{Q^{2\alpha-1}}, \quad (9)$$

one can obtain the second of the inequalities (5) with

$$\mu = \frac{1}{[\lambda(2\alpha-1)]^{\frac{1}{2\alpha-1}}} \left( 1 + \frac{1 + \ln \pi + 2 \ln(\sqrt{2} + \lambda(2\alpha-1)/2(1-\lambda))}{2\alpha-1} \right) < \left[ \pi e \left( \frac{\sqrt{2}}{\sqrt{\lambda}(2\alpha-1)} + \frac{1}{\sqrt{2}(1-\lambda)} \right)^2 \right]^{\frac{1}{2\alpha-1}}.$$

Putting

$$\lambda = \frac{1}{1 + \left(\frac{2\alpha-1}{4}\right)^{1/3}}$$

and noting that  $\frac{\pi e}{2} < 4.27$ , we obtain the required result.

The theorem is proved.

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*Note: Figure translations are in progress. See original paper for figures.*

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